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Quantization of 2D dilaton supergravity with matter

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Abstract: General $N = (1, 1)$ dilaton supergravity in two dimensions allows a background independent exact quantization of the geometric part, if these theories are formulated as specific graded Poisson-sigma models. The strategy developed for the bosonic case can be carried over, although considerable computational complications arise when the hamiltonian constraints are evaluated in the presence of matter. Nevertheless, the constraint structure is the same as in the bosonic theory. In the matterless case gauge independent nonlocal correlators are calculated non-perturbatively. They respect local quantum triviality and allow a topological interpretation.
In the presence of matter the ensuing nonlocal effective theory is expanded in matter loops. The lowest order tree vertices are derived and discussed, entailing the phenomenon of virtual black holes which essentially determine the corresponding S-matrix. Not all vertices are conformally invariant, but the S-matrix is invariant, as expected.
Finally, the proper measure for the 1-loop corrections is addressed. It is argued how to exploit the results from fixed background quantization for our purposes.

Keywords: BRST Quantization, 2D Gravity, Supergravity Models.

Contents

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1. Introduction

The traditional formulation of $N = (1,1)$ dilaton supergravity models in two dimensions is based upon superfields [1, 2]. In this approach the bosonic part of the supertorsion is set to zero and the fermionic component is assumed to be the one of flat supergravity [1]. In this way the Bianchi identities do not pose any problem, because they turn out to be fulfilled identically. Also, it is convenient to fix certain components in the superzweibein by a suitable gauge choice.

Although quite a number of interesting results has been obtained in this way, the complications due to the large number of auxiliary fields in the past have limited applications mostly to statements about the bosonic part of such theories for which ref. [2] is a key reference.

In the last years an alternative treatment of general *bosonic* dilaton theories has been very successful. It is based upon the local and global equivalence [3] of such theories to so-called Poisson-sigma models (PSMs) [4] of gravity where beside the zweibein also the spin connection appears as an *independent* variable together with new scalar fields ("target-space coordinates") on the 2D world sheet. Although those formulations possess non-vanishing bosonic torsion and thus a larger configuration space, the classical, as well as the quantum treatment, have been found to be much simpler. Based upon earlier work [5, 6] which had observed these features already before the advent of the PSM, many novel results could be obtained which seem to be inaccessible in practice in the usual dilaton field approach. They include the exact (background independent) quantization of the geometric part [6, 7, 8], the confirmation of the existence of "virtual Black Holes" without any further ad hoc assumptions [9, 10], the computation of the one-loop correction to the specific heat [11, 12] in the string inspired [13] dilaton Black Hole model [14] etc.[1] It should be pointed out that PSMs have attracted attention in string theory [18] due to the path integral interpretation [19] of Kontsevich's $\star$-product [20].

In order to benefit from these promising features also in the context of supergravity [21] it appeared natural to exploit *graded* Poisson-sigma models (gPSMs), where the target space is extended by a "dilatino" and the gauge fields comprise also the gravitino [22, 23]. Recently two of the present authors (L.B. and W.K.) succeeded [24, 25] in identifying a certain subclass of gPSMs with the "genuine" dilaton supergravity of ref. [2] when certain components of the superfield in the latter are properly expressed in terms of the fields appearing in the gPSM formulation. In the parlance of refs. [24, 25] the latter is called "minimal field supergravity" (MFS) in the following. MFS yielded the first complete solution (including fermions) for the 2D supergravity of ref. [2], the proper formulation of the super-pointparticle in the background of such a solution (and thus first nontrivial examples of "supergeodesics"), but also a full analysis of solutions retaining some supersymmetries, like BPS-states [26]. In the latter paper also minimal and non-minimal interactions with matter could be introduced at the gPSM level.

[1] A comprehensive review putting the results of the PSM-approach into the perspective of previous work on 2D gravity theories is [15]. Actually all known models (Jackiw-Teitelboim [16], spherically reduced Einstein gravity [17], the CGHS model [14] etc.) are special cases.

Spurred by the successful quantization in the PSM of bosonic gravity, in our present paper we carry out an analogous program for $N = (1,1)$ supergravity, minimally coupled to matter. This comprises as an essential intermediate step the evaluation of the constraint algebra. The strategy of bosonic theories can be carried over [6, 7, 8]: Geometry is integrated out first in a background-independent way. Certain boundary conditions for the geometry are fixed which selects a suitable background a posteriori. Finally, the effective theory is expanded perturbatively in matter loops, taking fully into account backreactions to each order in a self-consistent way.

This paper is organized as follows: section 2 briefly reviews the relations between gPSMs and MFS and the coupling to matter fields. In section 3 a hamiltonian analysis is performed. This is the necessary prerequisite for the construction of the BRST charge and the path integral quantization in section 4 employing a suitable gauge-fixing fermion. In the absence of matter the theory is found to be locally quantum trivial, although nontrivial nonlocal correlators exist. The lowest order matter vertices are derived in section 5 and their implications for the S-matrix are addressed. Section 6 is devoted to 1-loop corrections. The conclusions in 7 contain comments on generalizations to non-minimal coupling and to local self-interactions. Appendix A summarizes our notations and conventions. Some details which are relevant for the path integral, but which are somewhat decoupled from the main text, namely the treatment of boundary terms and the proof regarding the absence of ordering problems, can be found in appendix B.

2. Graded Poisson-sigma model and minimal field supergravity

2.1 Graded Poisson-sigma model

A general gPSM consists of scalar fields $X^I(x)$, which are themselves ("target space") coordinates of a graded Poisson manifold with Poisson tensor $P^{IJ}(X) = (-1)^{IJ+1}P^{JI}(X)$. The index I, in the generic case, may include commuting as well as anti-commuting fields.[2] In addition one introduces the gauge potential $A = \mathrm{d}X^I A_I = \mathrm{d}X^I A_{mI}(x)\,\mathrm{d}x^m$, a one form with respect to the Poisson structure as well as with respect to the 2D worldsheet. The gPSM action reads[3]

$$\mathcal{S}_{gPSM} = \int_{\mathcal{M}} \mathrm{d}X^I \wedge A_I + \frac{1}{2} P^{IJ} A_J \wedge A_I \,. \tag{2.1}$$

The Poisson tensor P^{IJ} must have vanishing Nijenhuis tensor (obeying a Jacobi-type identity with respect to the Schouten bracket related to the Poisson tensor as $\{X^I, X^J\} = P^{IJ}$)

$$P^{IL}\partial_L P^{JK} + \text{g-perm}\,(IJK) = 0 \,, \tag{2.2}$$

where the sum runs over the graded permutations. Due to (2.2) the action (2.1) is invariant under the symmetry transformations

$$\delta X^I = P^{IJ}\varepsilon_J \,, \qquad \delta A_I = -\mathrm{d}\varepsilon_I - (\partial_I P^{JK})\varepsilon_K\, A_J \,, \tag{2.3}$$

where the term $\mathrm{d}\epsilon_I$ in the second of these equations provides the justification for calling A_I "gauge fields".

[2]Details on the usage of different indices as well as other features of our notation are given in appendix A. For further details one should consult ref. [23, 28].

[3]If the multiplication of forms is evident the wedge symbol will be omitted subsequently.

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For a generic (g)PSM the commutator of two transformations (2.3) is a symmetry modulo the equations of motion. Only for P^{IJ} linear in X^I a closed (and linear) algebra is obtained, namely a graded Lie algebra; in this case (2.2) reduces to the Jacobi identity for the structure constants thereof. If the Poisson tensor has a non-vanishing kernel — the actual situation in any application to 2D (super-)gravity due to the odd dimension of the bosonic part of the tensor — there exist (one or more) Casimir functions $C(X)$ obeying

$$\{X^I, C\} = P^{IJ}\frac{\partial C}{\partial X^J} = 0\,, \tag{2.4}$$

which, when determined by the field equations

$$\mathrm{d}X^I + P^{IJ}A_J = 0\,, \tag{2.5}$$

$$\mathrm{d}A_I + \frac{1}{2}(\partial_I P^{JK})A_K A_J = 0\,, \tag{2.6}$$

are constants of motion.

In the most immediate application to 2D supergravity[4] the gauge potentials comprise the spin connection $\omega^a{}_b = \varepsilon^a{}_b\omega$, the dual basis e_a containing the zweibein and the gravitino ψ_α:

$$A_I = (A_\phi, A_a, A_\alpha) = (\omega, e_a, \psi_\alpha) \qquad X^I = (X^\phi, X^a, X^\alpha) = (\phi, X^a, \chi^\alpha)\,. \tag{2.7}$$

The fermionic components ψ_α ("gravitino") and χ^α ("dilatino") for $N = (1,1)$ supergravity are Majorana spinors. The scalar field ϕ will be referred to as "dilaton". The remaining bosonic target space coordinates X^a correspond to directional derivatives of the dilaton in the second order formulation presented below (cf. eq. 2.12). Local Lorentz invariance determines the ϕ-components of the Poisson tensor

$$P^{a\phi} = X^b\epsilon_b{}^a\,, \qquad P^{\alpha\phi} = -\frac{1}{2}\chi^\beta{\gamma_*}_\beta{}^\alpha\,, \tag{2.8}$$

and the supersymmetry transformation is encoded in $P^{\alpha\beta}$.

2.2 Recapitulation of dilaton gravity

In a purely bosonic theory, the only non-trivial component of the Poisson tensor is $P^{ab} = v\epsilon^{ab}$, where the locally Lorentz invariant "potential" $v = v(\phi, Y)$ describes different models ($Y = X^a X_a/2$). Evaluating (2.1) with that P^{ab} and $P^{a\phi}$ from (2.8), the action [4]

$$\mathcal{S}_{PSM} = \int_{\mathcal{M}} \left(\phi \mathrm{d}\omega + X^a De_a + \epsilon v\right) \tag{2.9}$$

is obtained. $\epsilon = \frac{1}{2}\epsilon^{ab}e_b \wedge e_a$ is the volume form and the covariant D in the torsion term is defined in (A.5). The most interesting models are described by potentials quadratic in X^a

$$v = Y\,Z(\phi) + V(\phi)\,. \tag{2.10}$$

[4]More complicated identifications of the 2D Cartan variables with A_I are conceivable [29].

All physically interesting 2D dilaton theories (spherically reduced Einstein gravity [17], the string inspired black hole [14], the Jackiw-Teitelboim model with $Z = 0$ and linear $V(\phi)$ [16], the bosonic part of the Howe model [1] etc.) are expressible by a potential of type (2.10). They allow the integration of the (single) Casimir function C in (2.4)

$$C = e^{Q(\phi)} Y + W(\phi)\,, \qquad Q(\phi) = \int_{\phi_1}^{\phi} \mathrm{d}\varphi Z(\varphi)\,, \qquad W(\phi) = \int_{\phi_0}^{\phi} \mathrm{d}\varphi e^{Q(\varphi)} V(\varphi)\,, \tag{2.11}$$

where e.g. in spherically reduced gravity C on-shell is proportional to the ADM-mass in the Schwarzschild solution. Its conservation $\mathrm{d}C = 0$ had been found previously in refs. [30].

The auxiliary variables X^a and the torsion-dependent part of the spin connection ω can be eliminated as they appear linear in the relevant equations of motion. Then the action reduces to the familiar generalized dilaton theory in terms of the dilaton field ϕ and the metric:

$$\mathcal{S}_{GDT} = \int \mathrm{d}^2 x \, \sqrt{-g} \left(\frac{1}{2} R\phi - \frac{1}{2} Z \partial^m \phi \partial_m \phi + V(\phi) \right). \tag{2.12}$$

Both formulations are equivalent at the classical [3, 31] as well as at the quantum level [7, 8].

For theories with non-dynamical dilaton ($Z = 0$ in (2.10)) a further elimination of ϕ is possible if the potential $V(\phi)$ is invertible. In this way one arrives at a theory solely formulated in terms of the metric $g_{mn} = e_m^a e_n^b \eta_{ab}$ [32].

2.3 Minimal field supergravity

A generic fermionic extension of the action (2.9) is obtained by the most general choice of $P^{a\alpha}$, $P^{\alpha\beta}$ and the fermionic extension of $P^{ab} = \epsilon^{ab}(v + \chi^2 v_2)$ solving (2.2). Here (2.8) and the bosonic potential v are a given input. This leads to an algebraic, albeit highly ambiguous solution of (2.2) with several arbitrary functions [23]. In addition, the fermionic extensions generically exhibit new singular terms. Also not all bosonic models permit such an extension for the whole range of their bosonic fields; sometimes even no extension is allowed.

As shown by two of the present authors [24], it is, nevertheless, possible to select "genuine" supergravity from this huge set of theories. This has been achieved by a generalization of the standard requirements for a "true" supergravity [21] to the situation, where deformations from the dilaton field ϕ are present. To this end the non-linear symmetry (2.3), which is closed on-shell only, is — in a first step — related to the more convenient (off-shell closed) algebra of hamiltonian constraints $G^I = \partial_1 X^I + P^{IJ}(X) A_{1J}$ discussed in detail in section 3. The hamiltonian obtained from (2.1) is a linear combination of these constraints [33, 6, 24]:

$$H = \int \mathrm{d}x^1 \; G^I A_{0I}\,. \tag{2.13}$$

In a second step a certain linear combination of the G^I, suggested by the ADM parametrization [34, 35, 24], maps the G^I algebra upon a deformed version of the superconformal algebra (deformed Neveu-Schwarz, resp. Ramond algebra). This algebra is appropriate to impose restrictions, which represent a natural generalization of the requirements from supergravity to theories deformed by the dilaton field. Translated into analytic restrictions onto the Poisson tensor they take the simple form[5]

$$\frac{\partial}{\partial X^{++}}(P^{+|-}, P^{+|++}, P^{+|--}) = 0\,, \qquad \frac{\partial}{\partial X^{--}}(P^{+|-}, P^{-|--}, P^{-|++}) = 0\,, \tag{2.14}$$

$$\partial_{++}P^{++|--} - 2\partial_{+}P^{+|--} = 0\,, \qquad \partial_{++}P^{++|-} - 2\partial_{+}P^{+|-} = 0\,, \tag{2.15}$$

whereby (2.15) can be derived from (2.14) together with (2.2).

It turned out that the subset of models allowed by these restrictions uniquely leads to the gPSM *supergravity* class of theories (called "minimal field supergravity", MFS, in our present paper) with the Poisson tensor[6] ($\chi^2 = \chi^\alpha \chi_\alpha$, $Y = X^a X_a/2$)

$$P^{ab} = \left(V + YZ - \frac{1}{2}\chi^2\left(\frac{VZ + V'}{2u} + \frac{2V^2}{u^3}\right)\right)\epsilon^{ab}\,, \tag{2.16}$$

$$P^{\alpha b} = \frac{Z}{4}X^a(\chi\gamma_a\gamma^b\gamma_*)^\alpha + \frac{iV}{u}(\chi\gamma^b)^\alpha\,, \tag{2.17}$$

$$P^{\alpha\beta} = -2iX^c\gamma_c^{\alpha\beta} + \big(u + \frac{Z}{8}\chi^2\big)\gamma_*{}^{\alpha\beta}\,, \tag{2.18}$$

where the three functions V, Z and the "prepotential" u depend on the dilaton field ϕ only. Besides the components of P^{IJ} already fixed according to (2.8), supergravity requires the existence of supersymmetry transformations, which are generated by the first term in (2.18). It is a central result of ref. [24] that $P^{\alpha\beta}$ must be of the form (2.18): The first term, which is not allowed to receive any deformations, is dictated by the flat limit of rigid supersymmetry. Furthermore, in order to satisfy the condition (2.2) V, Z and u must be related by

$$V(\phi) = -\frac{1}{8}((u^2)' + u^2 Z(\phi))\,. \tag{2.19}$$

Thus, starting from a certain bosonic model with potential

$$v = V + YZ \tag{2.20}$$

in (2.16), the only restriction is that it must be expressible in terms of a prepotential u by (2.19). This happens to be the case for all physically interesting theories [17, 14, 16, 1]. Inserting the Poisson tensor (2.8), (2.16)–(2.18) into equation (2.1) the ensuing action becomes (cf. eq. (A.5) for the definition of the respective covariant derivatives D)

$$\begin{aligned}\mathcal{S}_{MFS} = \int_{\mathcal{M}}\Bigg(&\phi \mathrm{d}\omega + X^a De_a + \chi^\alpha D\psi_\alpha + \epsilon\Big(V + YZ - \frac{1}{2}\chi^2\Big(\frac{VZ + V'}{2u} + \frac{2V^2}{u^3}\Big)\Big) + \\ &+ \frac{Z}{4}X^a(\chi\gamma_a\gamma^b e_b\gamma_*\psi) + \frac{iV}{u}(\chi\gamma^a e_a\psi) + iX^a(\psi\gamma_a\psi) - \frac{1}{2}\Big(u + \frac{Z}{8}\chi^2\Big)(\psi\gamma_*\psi)\Bigg)\,.\end{aligned} \tag{2.21}$$

As in case of the purely bosonic theory (2.9) the Poisson tensor has at least one (bosonic) Casimir function (2.11). In terms of the prepotential u it reads

$$C = e^Q(X^{++}X^{--} - \frac{1}{8}u^2 + \frac{1}{8}\chi^-\chi^+ C_\chi)\,, \tag{2.22}$$

[5]Here and in what follows light-cone coordinates are used, cf. eqs. (A.6)–(A.8).

[6]The constant $\tilde{u}_0$ in ref. [24] has been fixed as $\tilde{u}_0 = -2$. This is in agreement with standard supersymmetry conventions. f' denotes $df/d\phi$.

$$C_\chi = u' + \frac{1}{2}uZ = -4\frac{V}{u}\,. \tag{2.23}$$

If $C = 0$ the symmetric part of the Poisson tensor is degenerate and a second (fermionic) Casimir function $\tilde{c}$ emerges (cf. [23, 25]).

It has been proven in [25] that this class of supergravity models is equivalent to the superfield supergravity of Park and Strominger [2] upon elimination of auxiliary fields on both sides, when a certain linear combination of the gPSM gravitino and dilatino in MFS is identified with the gravitino of superfield supergravity (cf. section (5.2) of ref. [25] and footnote 19 below). The classical aspects of these models have been studied in some detail in refs. [25, 26].

2.4 Coupling of matter fields

This equivalence can be used to derive from the superspace construction the matter coupling for MFS models. For details of the calculations we refer to [26]. A supersymmetric matter multiplet consists of a real scalar field f and a Majorana spinor λ_α. In case of non-minimal coupling a coupling function $P(\phi)$ is introduced as well. Given the technical difficulties we restrict to minimal coupling ($P(\phi) \equiv 1$ in the notation of [26]) in the explicit calculations and generalizations will be commented upon the end. Then the matter action

$$\begin{aligned}\mathcal{S}_{(m)} = \int_{\mathcal{M}} \Big(&\frac{1}{2}\mathrm{d}f \wedge *\mathrm{d}f + \frac{i}{2}\lambda\gamma_a e^a \wedge *\mathrm{d}\lambda + i * (e_a \wedge *\mathrm{d}f) e_b \wedge *\psi\gamma^a\gamma^b\lambda + \\ &+ \frac{1}{4} * (e_b \wedge *\psi)\gamma^a\gamma^b e_a \wedge *\psi\lambda^2 \Big)\end{aligned} \tag{2.24}$$

is found to be invariant under local supersymmetry transformations.

3. Hamiltonian analysis

The primary goal of the present paper is to develop the systematics of the quantization of the action (2.21) together with matter couplings (2.24). The quantization is performed via a hamiltonian analysis introducing Poisson brackets. Though the appearance of fermions in both, geometry and matter part, lead to some technical complications the final result is seen to retain the structure already found in the bosonic case [33, 6, 7, 36]. For the purely geometrical part of the action the result of this analysis has been presented already in [24]. Nevertheless, a detailed formulation is given. For convenience many formulas are directly written in light-cone basis set out in appendix A.

Although we will proceed by analyzing MFS together with matter, an important remark about the range of Poisson tensors covered by our results should be made: The analysis of matterless (super-)geometry is valid for *any* graded Poisson tensor with local Lorentz invariance, i.e. whose components $P^{a\phi}$ and $P^{\alpha\phi}$ are determined by (2.8). Thus theories are covered as well that are not matterless supergravity in the sense of ref. [24]. The coupling of conformal matter fields according to the last section, however, is restricted only to the models of ref. [24] (i.e. the actions (2.21)). In what follows quantities evaluated

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from the gPSM part of the action are indicated by (g), while (m) labels the quantities from the matter action. This separation is possible as the matter action (2.24) does not contain derivatives acting on the MFS fields.

3.1 First class constraints

In the geometrical sector we define the canonical variables[7] and the first class primary constraints ($\approx$ means zero on the surface of constraints) from the lagrangian $L_{(g)}$ in (2.1) ($\dot{q}^I = \partial_0 X^I$) by

$$X^I = q^I\,, \qquad \bar{q}^I = (-1)^{I+1}\frac{\partial L}{\partial \dot{\bar{p}}_I} \approx 0\,, \tag{3.1}$$

$$\frac{\partial L}{\partial \dot{q}^I} = \frac{\partial L_{(g)}}{\partial \dot{q}^I} = p_I = A_{1I}\,, \qquad \bar{p}_I = A_{0I}\,. \tag{3.2}$$

From the hamiltonian density ($\partial_1 = \partial$)

$$H_{(g)} = \dot{q}^I p_I - L_{(g)} = \partial q^I \bar{p}_I - P^{IJ}\bar{p}_J p_I \tag{3.3}$$

the graded canonical equations

$$\frac{\partial H_{(g)}}{\partial p_I} = (-1)^I \dot{q}^I\,, \qquad \frac{\partial H_{(g)}}{\partial q^I} = -\dot{p}_I\,, \tag{3.4}$$

are consistent with the graded Poisson[8] bracket for functionals A and B

$$\begin{aligned}\{A, B'\} &= \int_{x''}\left[\left((-1)^{A\cdot I}\frac{\delta A}{\delta q''^I}\frac{\delta B'}{\delta p''_I} - (-1)^{I(A+1)}\frac{\delta A}{\delta p''_I}\frac{\delta B'}{\delta q''^I}\right) + (q \to \bar{q}, p \to \bar{p})\right] \\ &= \int_{x''}\left[\left(\left(\frac{\delta A}{\delta q''^i}\frac{\delta B'}{\delta p''_i} - \frac{\delta A}{\delta p''_i}\frac{\delta B'}{\delta q''^i}\right) + (-1)^A\left(\frac{\delta A}{\delta q''^\alpha}\frac{\delta B'}{\delta p''_\alpha} + \frac{\delta A}{\delta p''_\alpha}\frac{\delta B'}{\delta q''^\alpha}\right)\right) + \right.\\ &\quad \left. + (q \to \bar{q}, p \to \bar{p})\right]\,, \end{aligned} \tag{3.5}$$

where $(q \to \bar{q}, p \to \bar{p})$ indicates that the functional derivatives have to be performed for both types of variables, with and without bar. The primes indicate the dependence on primed world-sheet coordinates x, resp. x', x''. The hamiltonian density (3.3)

$$H_{(g)} = G^I_{(g)}\bar{p}_I \tag{3.6}$$

is expressed in terms of secondary constraints only:

$$\{\bar{q}^I, \int \mathrm{d}x^1\, H_{(g)}\} = G^I_{(g)} = \partial q^I + P^{IJ}p_J\,. \tag{3.7}$$

The extension to include conformal matter is straightforward. From the action (2.24) together with the matter fields

$$\mathsf{q} = f \qquad \mathsf{q}^\alpha = \lambda^\alpha \tag{3.8}$$

[7]The somewhat unusual association of gauge fields as "momenta" and of target space coordinate fields as "coordinates" is motivated in appendix B.1, where boundary conditions are discussed.

[8]This is the standard Poisson bracket, not to be confused with the Schouten bracket, associated to the Poisson tensor P^{IJ} in eqs. (2.2), (2.3).

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the canonical momenta[9]

$$\frac{\partial L_{(m)}}{\partial \dot{\mathfrak{q}}} = \mathfrak{p} = \frac{1}{e}\Big((p_{++}\bar{p}_{--} + p_{--}\bar{p}_{++})\partial\mathfrak{q} - 2p_{++}p_{--}\dot{\mathfrak{q}} + 2i\big(p_{++}p_{--}(\bar{p}_{-}\mathfrak{q}^{-} + \bar{p}_{+}\mathfrak{q}^{+}) - - \bar{p}_{++}p_{--}p_{+}\mathfrak{q}^{+} - p_{++}\bar{p}_{--}p_{-}\mathfrak{q}^{-}\big)\Big)\,, \tag{3.9}$$

$$\frac{\partial L_{(m)}}{\partial \dot{\mathfrak{q}}^{+}} = \mathfrak{p}_{+} = -\frac{1}{\sqrt{2}}p_{++}\mathfrak{q}^{+}\,, \tag{3.10}$$

$$\frac{\partial L_{(m)}}{\partial \dot{\mathfrak{q}}^{-}} = \mathfrak{p}_{-} = \frac{1}{\sqrt{2}}p_{--}\mathfrak{q}^{-} \tag{3.11}$$

are obtained. Analogous to (3.3) the hamiltonian density from the matter lagrangian in eq. (2.24) is defined as

$$H_{(m)} = \dot{\mathfrak{q}}\mathfrak{p} + \dot{\mathfrak{q}}^{+}\mathfrak{p}_{+} + \dot{\mathfrak{q}}^{-}\mathfrak{p}_{-} - L_{(m)}\,, \tag{3.12}$$

and the total hamiltonian density is the sum of (3.3) and (3.12). Obviously the definition in (3.12) implies a Poisson bracket for the matter fields with the same structure as given in (3.5). Especially for field monomials one finds

$$\{\mathfrak{q}, \mathfrak{p}'\} = \delta(x - x') \qquad \{\mathfrak{q}^{\alpha}, \mathfrak{p}'_{\beta}\} = -\delta^{\alpha}_{\beta}\delta(x - x')\,. \tag{3.13}$$

We do not provide the explicit form of the matter hamiltonian, as it can again be written in terms of secondary constraints:

$$H = G^{I}\bar{p}_{I}\,, \qquad G^{I} = G^{I}_{(g)} + G^{I}_{(m)}\,, \qquad \left\{\bar{q}^{I}, \int \mathrm{d}x^{1}\, H_{(m)}\right\} = G^{I}_{(m)}\,. \tag{3.14}$$

The explicit expressions for the matter part of the secondary constraints read:

$$G^{++}_{(m)} = -\frac{1}{4p_{++}}(\partial\mathfrak{q} - \mathfrak{p})^{2} + \frac{1}{\sqrt{2}}\mathfrak{q}^{+}\partial\mathfrak{q}^{+} + \frac{i}{p_{++}}(\partial\mathfrak{q} - \mathfrak{p})p_{+}\mathfrak{q}^{+}\,, \tag{3.15}$$

$$G^{--}_{(m)} = \frac{1}{4p_{--}}(\partial\mathfrak{q} + \mathfrak{p})^{2} - \frac{1}{\sqrt{2}}\mathfrak{q}^{-}\partial\mathfrak{q}^{-} - \frac{i}{p_{--}}(\partial\mathfrak{q} + \mathfrak{p})p_{-}\mathfrak{q}^{-}\,, \tag{3.16}$$

$$G^{+}_{(m)} = i(\partial\mathfrak{q} - \mathfrak{p})\mathfrak{q}^{+}\,, \tag{3.17}$$

$$G^{-}_{(m)} = -i(\partial\mathfrak{q} + \mathfrak{p})\mathfrak{q}^{-}\,. \tag{3.18}$$

3.2 Second class constraints

As the kinetic term of the matter fermion λ is first order only, this part of the action leads to constraints as well. From (3.10) and (3.11) the usual primary second-class constraints are deduced:

$$\Psi_{+} = \mathfrak{p}_{+} + \frac{1}{\sqrt{2}}p_{++}\mathfrak{q}^{+} \approx 0\,, \tag{3.19}$$

$$\Psi_{-} = \mathfrak{p}_{-} - \frac{1}{\sqrt{2}}p_{--}\mathfrak{q}^{-} \approx 0\,. \tag{3.20}$$

[9] In what follows, all expressions are written in terms of canonical variables. The determinant $\sqrt{-g} = e$ in these variables reads $e = p_{--}\bar{p}_{++} - p_{++}\bar{p}_{--}$.

These second class constraints are treated by substituting the Poisson bracket by the "Dirac bracket" [37, 38, 39]

$$\{f,g\}^* = \{f,g\} - \{f,\Psi_\alpha\}C^{\alpha\beta}\{\Psi_\beta,g\}\,, \tag{3.21}$$

where

$$C^{\alpha\beta}C_{\beta\gamma} = \delta^\alpha_\gamma\,, \qquad C_{\alpha\beta} = \{\Psi_\alpha,\Psi_\beta\}\,. \tag{3.22}$$

From (3.19) and (3.20) together with the definition of the canonical bracket the matrix $C_{\alpha\beta}$ follows as

$$C_{\alpha\beta} = \sqrt{2}\begin{pmatrix} -p_{++} & 0 \\ 0 & p_{--}\end{pmatrix}. \tag{3.23}$$

In particular, the Dirac bracket among two field monomials has the non-trivial components

$$\{\mathfrak{q}^+,\mathfrak{q}'^+\}^* = \frac{1}{\sqrt{2}p_{++}}\delta(x-x')\,, \qquad \{\mathfrak{q}^-,\mathfrak{q}^-\}^* = -\frac{1}{\sqrt{2}p_{--}}\delta(x-x')\,, \tag{3.24}$$

$$\{\mathfrak{p}_+,\mathfrak{p}'_+\}^* = \frac{p_{++}}{2\sqrt{2}}\delta(x-x')\,, \qquad \{\mathfrak{p}_-,\mathfrak{p}'_-\}^* = -\frac{p_{--}}{2\sqrt{2}}\delta(x-x')\,, \tag{3.25}$$

$$\{\mathfrak{q}^+,\mathfrak{p}'_+\}^* = -\frac{1}{2}\delta(x-x')\,, \qquad \{\mathfrak{q}^-,\mathfrak{p}'_-\}^* = -\frac{1}{2}\delta(x-x')\,, \tag{3.26}$$

$$\{q^{++},\mathfrak{p}'_+\}^* = -\frac{\mathfrak{q}^+}{2\sqrt{2}}\delta(x-x')\,, \qquad \{q^{--},\mathfrak{p}'_-\}^* = \frac{\mathfrak{q}^-}{2\sqrt{2}}\delta(x-x')\,, \tag{3.27}$$

$$\{q^{++},\mathfrak{q}'^+\}^* = -\frac{\mathfrak{q}^+}{2p_{++}}\delta(x-x')\,, \qquad \{q^{--},\mathfrak{q}'^-\}^* = -\frac{\mathfrak{q}^-}{2p_{--}}\delta(x-x')\,, \tag{3.28}$$

and all other brackets are unchanged.

3.3 Dirac bracket algebra of secondary constraints

The algebra of hamiltonian constraints is the result of a straightforward but tedious calculation. We make a few remarks on important observations therein.

As the $G^I_{(m)}$ are independent of q^I, the p_I in (3.7) commutes trivially within $\{G^I_{(g)},G^J_{(m)}\}^*$. There are, however, non-trivial commutators with q^I: (3.15) and (3.16) depend on the coordinates $p_\pm$ and all constraints (3.15)–(3.18) depend on the fermionic matter field $\mathfrak{q}^\pm$, which has a non-vanishing Dirac bracket with $q^{\pm\pm}$ (3.28). Indeed, after a straightforward calculation one arrives at

$$\{G^{++}_{(m)},q'^{++}\}^* = \frac{1}{p_{++}}G^{++}_{(m)} + \frac{1}{2p_{++}^2}p_+G^+_{(m)}\delta(x-x')\,, \tag{3.29}$$

$$\{G^{--}_{(m)},q'^{--}\}^* = \frac{1}{p_{--}}G^{--}_{(m)} + \frac{1}{2p_{--}^2}p_-G^-_{(m)}\delta(x-x')\,, \tag{3.30}$$

$$\{G^{++}_{(m)},q'^+\}^* = \frac{1}{p_{++}}G^+_{(m)}\delta(x-x')\,, \qquad \{G^{--}_{(m)},q'^-\}^* = \frac{1}{p_{--}}G^-_{(m)}\delta(x-x')\,, \tag{3.31}$$

$$\{G^+_{(m)},q'^{++}\}^* = \frac{1}{2p_{++}}G^+_{(m)}\delta(x-x')\,, \qquad \{G^-_{(m)},q'^{--}\}^* = \frac{1}{2p_{--}}G^-_{(m)}\delta(x-x')\,, \tag{3.32}$$

which are useful relations in the calculation of the "mixed commutators" $\{G^I_{(g)},G^J_{(m)}\}^*$. Also notice that $\frac{1}{p_{++}}G^+_{(m)}$ is part of $G^{++}_{(m)}$ as well.

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However, the most important relations involved in the calculation including the matter extension are the supergravity restrictions (2.14) and (2.15). Also the fact that the "supersymmetry transformation" (the $P^{+|+}$, resp. $P^{-|-}$ part of the Poisson tensor) is model independent leads to cancellations between terms from the "mixed" commutators together with terms from the purely matter part $\{G^I_{(m)}, G^J_{(m)}\}^*$.

Putting the pieces together the final result

$$\{G^I, G'^J\}^* = G^K C_K{}^{IJ} \delta(x - x')\,, \qquad C_K{}^{IJ} = -\partial_K P^{IJ} \tag{3.33}$$

is obtained. It is an important confirmation of the construction of the last section that the result as expected from bosonic gravity is reproduced: Minimally coupled conformal matter does not change the structure functions of the constraint algebra as compared to the matterless case. Indeed, when Dirac brackets are used, the result of [24] is found up to the substitution $G^I_{(g)} \to G^I = G^I_{(g)} + G^I_{(m)}$.

Eq. (3.33) is the main result of this section and we would like to conclude with some comments on its algebraic structure. Beside the constraints G^I, the r.h.s. of this relation depends on the canonical coordinates q^I. To study the closure of this non-linear algebra, at least that coordinate must be part of the algebra as well. In super-geometry it turns out that the algebra of $G^I_{(g)}$ and q^I closes as [24]

$$\{G^I_{(g)}, q'^J\} = -P^{IJ} \delta(x - x')\,. \tag{3.34}$$

Therefore the system $(G^I_{(g)}, q^I)$ defines a graded finite W algebra. It simplifies to a graded Lie-algebra if and only if the Poisson tensor is linear in the target-space coordinates. As long as only super-geometry is considered, the relations (3.33) and (3.34) hold for any Poisson tensor.

In the presence of matter interaction (2.24), eq. (3.34) must be replaced by $\{G^I, q'^J\}^*$, and the complete algebra is obtained by adding (3.33) and (3.29)–(3.32). Notice that on the r.h.s. of $\{G^I, q'^J\}^*$ the terms $G^I_{(g)}$ and $G^I_{(m)}$ never appear together, because $P^{++|++} = P^{++|+} = 0$. As the r.h.s. of (3.29)–(3.32) depend on p_J as well, the algebra no longer closes together with q^I, but the former coordinate must be part of the algebra as well. But since [24]

$$\{G^I_{(g)}, p'_J\} = (-1)^I \partial\delta(x - x')\delta^I_J + (-1)^{IJ} \partial_J P^{IK} p_K \delta(x - x')\,, \tag{3.35}$$

this is not a W algebra, as expected from the analogous situation in bosonic case.

4. Quantization

4.1 Ghosts and gauge fixing

The construction outlined in this section closely follows ref. [39]. However, some details of the grading are different, as the canonical index position in that book is different from the conventions used here. Our constraints have upper indices and thus the same applies to the anti-ghosts to be introduced in relation to the constraints of our system:

$$\text{primary constraints}: \; b_I\,,\; p_b^I \tag{4.1}$$

$$\text{secondary constraints}: \; c_I\,,\; p_c^I\,. \tag{4.2}$$

The brackets between ghosts and anti-ghosts are defined conveniently as

$$[b_I, p_b^J] = -(-1)^{(I+1)(J+1)}[p_b^J, b_I] = \delta_I^J \,, \qquad [c_I, p_c^J] = -(-1)^{(I+1)(J+1)}[p_c^J, c_I] = \delta_I^J \,. \tag{4.3}$$

To first order in homological perturbation theory the BRST charge Ω follows straightforwardly:

$$\Omega = \bar{q}^I b_I + G^I c_I - \frac{1}{2}(-1)^I p_c^K \partial_K P^{IJ} c_J c_I \,. \tag{4.4}$$

Note that $P^{IJ} c_J c_I = (-1)^{I+J} P^{JI} c_I c_J$.

For the special case $G^I = G^I_{(g)}$ the BRST charge (4.4) is found to be nilpotent, simply as a consequence of the graded Jacobi identity[10] of P^{IJ} (2.2). But even for the constraints involving $G^I_{(m)}$ the homological perturbation theory stops at this order: For models with $Z = 0$ in (2.21) all relevant brackets $[G^I, \partial_L P^{JK}]^*$ vanish trivially, for the general case the restrictions (2.15) guarantee nilpotency.

As the hamiltonian vanishes on the constraint surface it simply becomes

$$H_{gf} = \{\Omega, \Psi\} \tag{4.5}$$

for some gauge fixing fermion Ψ. Following [27] a multiplier gauge $\Psi = p_c^I a_I$ is used with constant a_I, for which the gauge-fixed hamiltonian reads

$$\{\Omega, \Psi\} = -G^I a_I + (-1)^K p_c^I \partial_I P^{JK} c_K a_J \,. \tag{4.6}$$

The canonical equations of the ghosts are

$$\frac{\delta H_{gf}}{\delta p_c^I} = \dot{c}_I \,, \qquad \frac{\delta H_{gf}}{\delta c_I} = (-1)^I \dot{p}_c^I \,, \tag{4.7}$$

and the corresponding gauge-fixed lagrangian reads

$$L_{gf} = \dot{\bar{q}}^I \bar{p}_I + \dot{q}^I p_I + \dot{\mathfrak{q}}\mathfrak{p} + \dot{\mathfrak{q}}^\alpha \mathfrak{p}_\alpha + p_c^I \dot{c}_I + p_b^I \dot{b}_I - H_{gf} \,. \tag{4.8}$$

Even in such a simple gauge the construction of the effective lagrangian and the subsequent integration over its variables in the path integral can lead to lengthy equations. Thus we restrict the explicit calculations to the simplest possible gauge, which is not inconsistent with physical requirements such as the non-degeneracy of the bosonic metric:[11] $a_I = -i\delta_I{}^{++}$. This entails the Eddington-Finkelstein form of the bosonic line element.

The gauge-fixed lagrangian (4.5)–(4.8) with this choice becomes

$$\begin{aligned} L_{gf} = {} & \dot{\bar{q}}^I \bar{p}_I + \dot{q}^I p_I + \dot{\mathfrak{q}}\mathfrak{p} + \dot{\mathfrak{q}}^\alpha \mathfrak{p}_\alpha + p_c^I \dot{c}_I + p_b^I \dot{b}_I - i\partial q^{++} - iP^{++|J} p_J + \frac{i}{4p_{++}}(\partial\mathfrak{q} - \mathfrak{p})^2 - \\ & - \frac{i}{\sqrt{2}}\mathfrak{q}^+ \partial \mathfrak{q}^+ + \frac{1}{p_{++}}(\partial\mathfrak{q} - \mathfrak{p}) p_+ \mathfrak{q}^+ + i(-1)^K p_c^I \partial_I P^{++|K} c_K \,, \end{aligned} \tag{4.9}$$

which will be the starting point of our calculations in the path integral formalism.

[10] Without coupling of matter fields $\Omega^2 = 0$ does not depend on the specific form of the Poisson tensor in (2.16)–(2.18).

[11] Notice that according to the notation (A.7) p_{++} is purely imaginary. Also, one component (q^+, $\mathfrak{q}^-$) of a spinor is real, while the other (q^-, $\mathfrak{q}^+$) is imaginary.

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4.2 Path integral formalism

Before formally integrating out some of the fields by the path integral formalism it is worthwhile to check that no problems arise due to ordering ambiguities hidden in the definition of its measure (which are typical for nonlinear interactions, cf. e.g. [40]) and that boundary contributions are treated with care. Since these considerations are somewhat tangential to the main topics of this paper, they are relegated to appendix B, where it is shown that problems of this type do not arise.

The generating functional of Green functions that follows from the result of the previous section has to be integrated over all physical fields and all ghosts. Together with sources $j_{q\,I}$, ${j_p}^I$ for the geometrical variables and J, J_α for the matter fields it reads:

$$\begin{aligned} \mathcal{W}[j_{q\,I}, {j_p}^I, J, J_\alpha] = \int \mathcal{D}(q^I, p_I, \bar{q}^I, \bar{p}_I, \mathfrak{q}, \mathfrak{p}, \mathfrak{q}^\alpha, \mathfrak{p}_\alpha, c_I, p_c^I, b_I, p_b^I) \times \\ \times \exp\left(i \int \mathrm{d}^2 x \, \left(L_{gf} + q^I j_{q\,I} + {j_p}^I p_I + \mathfrak{q} J + \mathfrak{q}^\alpha J_\alpha \right) \right) . \end{aligned} \tag{4.10}$$

The gauge-fixed hamiltonian, being independent of $\bar{q}^I$, $\bar{p}_I$, b_I and p_b^I, allows a trivial integration of all these fields. As the remaining ghosts appear at most bi-linearly in the action they can be integrated over, which leads to the super-determinant

$$\mathrm{sdet} {M_I}^J = \mathrm{sdet}\big({\delta_I}^J \partial_0 - i\partial_I P^{++|J} \big) . \tag{4.11}$$

The integration of $\mathfrak{p}_\alpha$ by means of the constraint (3.19) and (3.20) is trivial as well, while the bosonic momentum $\mathfrak{p}$ of matter can be integrated after the quadratic completion

$$\begin{aligned} L_{(m)} &= \dot{\mathfrak{q}}\mathfrak{p} + \dot{\mathfrak{q}}^\alpha \mathfrak{p}_\alpha + \frac{i}{4p_{++}}(\partial\mathfrak{q} - \mathfrak{p})^2 - \frac{i}{\sqrt{2}}\mathfrak{q}^+\partial\mathfrak{q}^+ + \frac{1}{p_{++}}(\partial\mathfrak{q} - \mathfrak{p})p_+\mathfrak{q}^+ \\ &= \frac{i}{4p_{++}}\big(\mathfrak{p} - \partial\mathfrak{q} + 2ip_+\mathfrak{q}^+ - 2ip_{++}\dot{\mathfrak{q}}\big)^2 + ip_{++}\dot{\mathfrak{q}}^2 + \dot{\mathfrak{q}}(\partial\mathfrak{q} - 2ip_+\mathfrak{q}^+) - \\ &\quad - \frac{1}{\sqrt{2}}p_{++}\dot{\mathfrak{q}}^+\mathfrak{q}^+ + \frac{1}{\sqrt{2}}p_{--}\dot{\mathfrak{q}}^-\mathfrak{q}^- - \frac{i}{\sqrt{2}}\mathfrak{q}^+\partial\mathfrak{q}^+ \end{aligned} \tag{4.12}$$

and a suitable shift in this variable. After a gaussian integration in the shifted $\mathfrak{p}$ this leads to a new effective lagrangian linear in p_I, which allows to integrate these fields as well. The resulting functional δ-functions finally determine the q^I. It is advantageous to proceed through these steps separately for the matterless and for the full theory.

4.3 Matterless case

4.3.1 Integrating out geometry

After integrating the remaining ghosts the lagrangian

$$L_{\text{eff}} = \dot{q}^I p_I - i\partial q^{++} - iP^{++|J}p_J + q^I j_{q\,I} + {j_p}^I p_I \tag{4.13}$$

is obtained. The p_I only appear *linearly* in (4.13) which upon integration yields five functional δ functions. The arguments of the latter imply the solution of a system of as many coupled differential equations, which for the Poisson tensor (2.8), (2.16)–(2.18) become ($W = -e^Q u^2/8$, cf. eq. (2.11))

$$\dot{q}^{\phi} = -iq^{++} - j_p{}^{\phi}\,, \tag{4.14}$$
$$\dot{q}^{++} = -j_p{}^{++}\,, \tag{4.15}$$
$$\dot{q}^{--} = i\Big(e^{-Q}W' + q^{++}q^{--}Z + \frac{1}{2\sqrt{2}}q^- q^+ e^{-Q/2}(\sqrt{-W})''\Big) - j_p{}^{--}\,, \tag{4.16}$$
$$\dot{q}^{+} = -j_p{}^{+}\,, \tag{4.17}$$
$$\dot{q}^{-} = -i\big(e^{-Q/2}(\sqrt{-W})'q^+ - \frac{1}{2}Zq^{++}q^-\big) - j_p{}^{-}\,, \tag{4.18}$$

and can be used to perform the final q^I integration. As expected, this exactly cancels the super-determinant (4.11). The solutions of (4.14)–(4.18) are denoted by B^I and may formally be written as

$$B^{\phi} = \hat{q}^{\phi} + \partial_0^{-1}A^{\phi}\,, \tag{4.19}$$
$$B^{++} = \hat{q}^{++} + \partial_0^{-1}A^{++}\,, \qquad B^{--} = e^{-D^{--}}\big(\hat{q}^{--} + \partial_0^{-1}A^{--}\big)\,, \tag{4.20}$$
$$B^{+} = \hat{q}^{+} + \partial_0^{-1}A^{+}\,, \qquad B^{-} = e^{-D^{--}/2}\big(\hat{q}^{-} + \partial_0^{-1}A^{-}\big)\,, \tag{4.21}$$

with

$$A^{\phi} = -iB^{++} - j_p{}^{\phi}\,, \tag{4.22}$$
$$A^{++} = -j_p{}^{++}\,, \tag{4.23}$$
$$A^{--} = e^{D^{--}}\left[ie^{-Q}W' + \frac{i}{2\sqrt{2}}e^{-Q/2}(\sqrt{-W})''B^-B^+ - j_p{}^{--}\right], \tag{4.24}$$
$$A^{+} = -j_p{}^{+}\,, \tag{4.25}$$
$$A^{-} = -e^{D^{--}/2}\left[ie^{-Q/2}(\sqrt{-W})'B^+ + j_p{}^{-}\right]. \tag{4.26}$$

Here the $\hat{q}^I(x^1)$ are integration constants $\partial_0\hat{q}^I = 0$, ∂_0^{-1} denotes a properly defined Green function and $D^{--} = i\partial_0^{-1}(ZB^{++})$. With these definitions the solution for the matterless path integral finally becomes

$$\mathcal{W}[j_p{}^I, j_{q_I}] = \exp iL^0_{\text{eff}}\,, \qquad L^0_{\text{eff}} = \int \mathrm{d}^2x\,\left(B^I j_{q_I} + \tilde{L}^0(j_p{}^I, B^I)\right). \tag{4.27}$$

$\tilde{L}^0$ are the so-called "ambiguous terms". An expression of this type is generated by the integration constants $G_I(x^1)$ from the term $\int\mathrm{d}x^0\,\int\mathrm{d}y^0\,(\partial_0^{-1}A^I)j_{q_I}$ (cf. section 7 of [15] and references therein). Thus, the ambiguous terms are

$$\tilde{L}^0 = A^I g_I\,. \tag{4.28}$$

with A^I given by (4.22)–(4.26). The physical meaning of the integrations constants g_I can be seen from the expectation values $< p_I >$ which have to be adjusted in accordance with the boundary conditions imposed on these fields. For instance, the expectations values $< \psi_1^{\pm} >$ are required to vanish asymptotically and hence the corresponding constants $g_{\pm}$ have to be set to zero. Then the matter vertices to be calculated below will be generated solely by A^{--}, just as in the bosonic case.

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4.3.2 Effective action and local quantum triviality

It is instructive to calculate the Legendre transform of L^0_{eff}, viz. the effective action

$$\Gamma\left(\langle q^I\rangle,\langle p_I\rangle\right) := L^0_{\text{eff}}\left(j_{q_I},{j_p}^I\right) - \int \mathrm{d}^2x\ \left(\langle q^I\rangle j_{q_I} + {j_p}^I\langle p_I\rangle\right)\,, \tag{4.29}$$

in terms of the mean fields

$$\langle p_I\rangle := \frac{\overrightarrow{\delta}}{\delta {j_p}^I} L^0_{\text{eff}}\Bigg|_{j=0}\,, \qquad \langle q^I\rangle := L^0_{\text{eff}}\frac{\overleftarrow{\delta}}{\delta j_{q_I}}\Bigg|_{j=0} = B^I\big|_{j_p=0}\,. \tag{4.30}$$

To ensure that the mean fields attain their classical values without supersymmetry signs, left variation is used in the left formula of (4.30) and right variation in the right one. Plugging in the previous results (4.27), (4.28) of this section immediately yields

$$\Gamma\left(\langle q^I\rangle,\langle p_I\rangle\right) = \int \mathrm{d}^2x\ \left(A^I g_I - {j_p}^I\langle p_I\rangle\right), \tag{4.31}$$

where the sources ${j_p}^I$ have to be expressed in terms of the (classical) target space coordinates and their first derivatives by virtue of (4.14)-(4.18). Consequently, the effective action (4.31) turns out to be nothing but the gauge fixed *classical* action (4.13) in terms of the mean fields plus the nontrivial boundary contributions[12]

$$\pm \int_{\partial M} \mathrm{d}x^1 \left(e^{D^{--}} B^{--} g_{--} - e^{D^{--}/2} B^- g_-\right). \tag{4.32}$$

Requiring asymptotically vanishing fermion fields (cf. section 5.1 below eq. (5.4)), $g_- = 0$ and the second term vanishes. The sign in front of (4.32) depends on the sign of the outward pointing normal vector of the boundary surface ∂M. Thus, local quantum triviality generalizes from the bosonic case [7] to the supersymmetric one.

4.3.3 Nonlocal quantum correlators

Before switching on matter interactions we would like to address a further point, which already has been observed (but not discussed) in the bosonic case [41]: although all canonical variables acquire their classical values as their expectation values, the correlators between two or more variables need not decompose into a product of classical values — the latter feature being very well-known from generic quantum field theory. In particular, we obtain

$$\left\langle q^I q^J\right\rangle = \left\langle q^I\right\rangle\left\langle q^J\right\rangle, \qquad \langle p_I p_J\rangle \neq \langle p_I\rangle\,\langle p_J\rangle\,, \tag{4.33}$$

and according to (4.19)–(4.21)

$$\left\langle q^I(x) p_J(y)\right\rangle = \left\langle q^I(x)\right\rangle \langle p_J(y)\rangle + \frac{\delta}{i\delta {j_p}^J(y)} B^I(x)\,. \tag{4.34}$$

[12] The boundary terms result from $A^I g_I$, so actually there are five of them. However, three of them are rather trivial, as can be checked most easily by expressing the sources in (4.22)-(4.26) in terms of the classical target space coordinates using again (4.14)-(4.18).

Eq. (4.33) e.g. entails that the product of two zweibein components is not necessarily equal to the product of the vacuum expectation values thereof. Note that in order to create nontrivial correlators of this type the dependence on external sources must be non-linear. This rules out many of the possible correlators. Among the expressions (4.34) with purely bosonic field content

$$\langle X^a(x)e_{a1}(y)\rangle = \left\langle q^{++}(x)p_{++}(y) + q^{--}(x)p_{--}(y)\right\rangle \tag{4.35}$$

is the only Lorentz invariant contribution that receives non-perturbative quantum corrections. In the coincidence limit $x = y$ (4.35) reduces to a 1-form times the Killing norm.

Even simpler are correlators of fermionic operators. According to our boundary conditions employed below (cf. section 5.1)

$$\left\langle \chi^{\pm}\right\rangle = 0 = \langle \psi_{\pm}\rangle \tag{4.36}$$

occurs. This implies that all correlators involving one or more insertions of a dilatino or a gravitino vanish, with the notable exception of $\langle\chi^+(x)\psi_{+1}(y)\rangle$. This one turns out to be especially useful to study the systematics of the non-perturbative quantum effects: In contrast to (4.35) the classical ("dominant") value vanishes from (4.36) and consequently any nontrivial contribution stems exclusively from quantum effects.

To calculate this correlator explicitly we should impose certain prescriptions for the integration constants in (4.34). With $\partial_0^{-1}{}_{xz}f(z) := \int_y^x f(z)\mathrm{d}z$ and $\theta(0) = 1/2$ it becomes[13] the field independent expression

$$\left\langle \chi^+(x)\psi_{+1}(y)\right\rangle = \frac{\delta}{i\delta {j_p}^+(y)}B^+(x) = \frac{1}{i}\left(\theta(y^0 - x^0) - \frac{1}{2}\right)\delta(x^1 - y^1)\,, \tag{4.37}$$

where the lower boundary y is chosen such that in the coincidence limit $x^0 = y^0$ of (4.37) the correlator vanishes. This correlator is purely nonlocal in (x^0, y^0), but local in (x^1, y^1) and obeys the following identities (we recall that $\psi_{\pm 0} = 0$):

$$\left\langle \chi^+(x)\psi_{+m}(y)\right\rangle = \left\langle \chi^-(x)\psi_{-m}(y)\right\rangle\,, \qquad \left\langle \chi^{\pm}(y)\psi_{\pm m}(x)\right\rangle = \left\langle \chi^{\pm}(x)\psi_{\pm m}(y)\right\rangle^{\dagger}\,. \tag{4.38}$$

Therefore, the matrix

$$\begin{pmatrix} \langle\chi^+(x)\chi_+(y)\rangle & \langle\psi_1^+(x)\chi_+(y)\rangle \\ \langle\chi^+(x)\psi_{+1}(y)\rangle & \langle\psi_1^+(x)\psi_{+1}(y)\rangle \end{pmatrix} \tag{4.39}$$

has non-vanishing entries in the off-diagonal only. There is a mixed gravitino-dilatino correlator which is completely independent of the geometric properties encoded in the bosonic potentials u and Z. While the pseudo-scalar expression

$$\langle\chi^\alpha(x)\gamma_*\psi_{\alpha m}(y)\rangle = \left\langle \chi^+(x)\psi_{+m}(y)\right\rangle - \left\langle \chi^-(x)\psi_{-m}(y)\right\rangle = 0 \tag{4.40}$$

[13] Other prescriptions are possible, of course, but eventual ambiguities in the definitions are fixed by the (physical) requirement that in the coincidence limit the correlator vanishes, because after all the theory is *locally* quantum trivial. The choice $\theta(0) = 1/2$ has the advantage that the identity $\theta(a - b) = 1 - \theta(b - a)$ can be used even if $a = b$. The results of this section on nonlocal correlators ($a \neq b$) do not depend on the choice of the prescription.

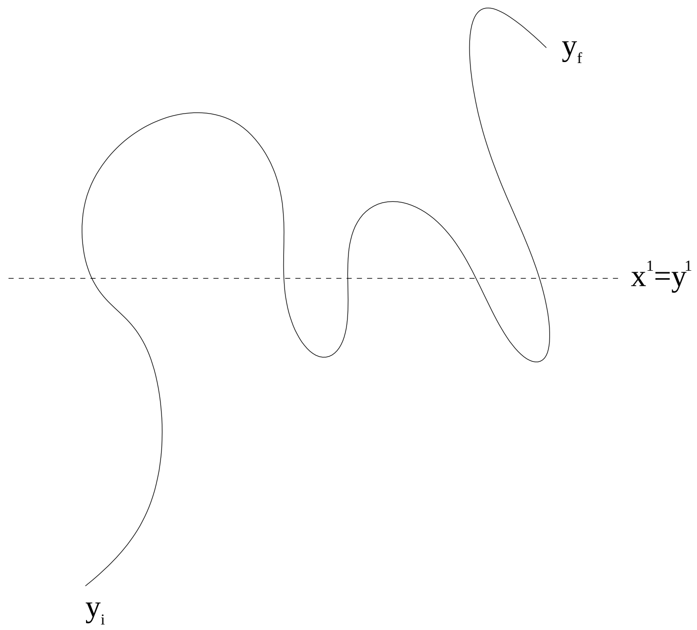

Figure 1: A typical path $\mathcal{P}$ with $n^r_+ = 3$ and $n^r_- = 2$. It is supposed that the point x is on the left side of this graph, i.e. $x^0 < y^0$ for all y lying on $\mathcal{P}$. Thus, $\mathcal{K} = 3 - 2 - 0 + 0 = 1$.

vanishes, the scalar one is non-vanishing ($\varepsilon(a-b) := \theta(a-b) - \theta(b-a)$):

$$\langle \chi^\alpha(x)\psi_{\alpha m}(y)\rangle = \frac{1}{i}\varepsilon(y^0 - x^0)\delta(x^1 - y^1)\delta_{1m}\,. \tag{4.41}$$

In order to get a gauge independent[14] expression one has to consider the correlator integrated over a path $\mathcal{P}$,

$$\mathcal{K} := i\left\langle \int_{\mathcal{P}} \mathrm{d}y^m \chi^\alpha(x)\psi_{\alpha m}(y)\right\rangle = \int_{\mathcal{P}} \mathrm{d}y^1 \varepsilon(y^0 - x^0)\delta(x^1 - y^1) = (n^r_+ - n^r_- - n^l_+ + n^l_-)\,. \tag{4.42}$$

The natural numbers $n^{rl}_\pm$ count how often the line $y^1 = x^1$ is intersected from below $(+)$ and from above $(-)$ by the path $\mathcal{P}$. The additional label r, l contains the information whether the intersection point lies on the left or the right hand side of x ("left" means $x^0 > y^0$). This can be visualized most easily by plotting the path in a cartesian diagram with x-axis $(y^0 - x^0)$ and y-axis $(y^1 - x^1)$. Any intersection of the path $\mathcal{P}$ with the y-axis from an even (odd) quadrant to an odd (even) one contributes $+1$ (-1) to $\mathcal{K}$. Intersections through the origin do not contribute. The integrated correlator $\mathcal{K}$ takes values in $\mathbb{Z}$. Obviously, $\mathcal{K}$ is independent of the choice of integration constants as long as the Wilson loop does not pass through the origin (cf. footnote 13).

For strip-like topology a typical open path from the point y_i to y_f is displayed in figure 1. An example for a closed path with nontrivial winding is given in figure 2. Obviously,

[14] The result of our construction resembles a Wilson loop. While we have some reservations regarding the observability of a generic Wilson loop (as opposed to certain quantities derived from it like the anomalous dimension), cf. e.g. [42], in the present context the attribute "gauge independent" is justified because the correlator does not receive any contributions from renormalization as our derivation has been an exact one rather than perturbative.

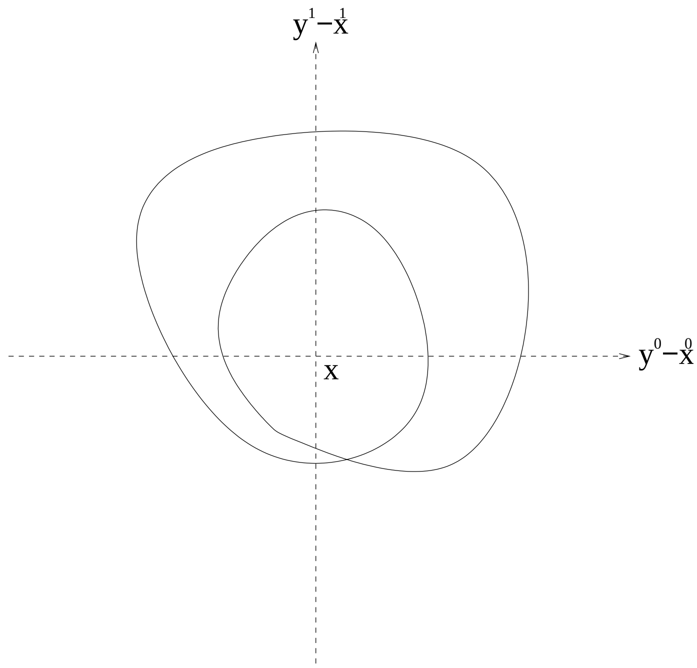

Figure 2: A closed path winding around the point x with $\mathcal{K} = \pm 4$. The sign depends on the orientation and it is positive for counter clockwise orientation.

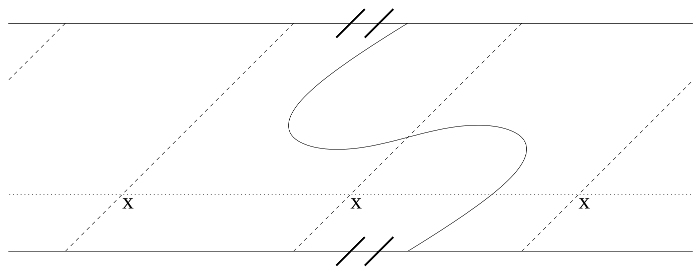

Figure 3: A nontrivial closed path for cylindrical topology with $\mathcal{K} = \pm 1$, depending again on the orientation of the loop.

$\mathcal{K}/2$ can be interpreted as winding number around the point x. For cylindrical topology (figure 3) further complications are possible: if $\mathcal{K}/2$ is again interpreted as winding number this implies that cylindrical topology allows for a fractional (half-integer) winding number. The upper and lower boundaries have to be identified in that graph. The non-compact direction corresponds to "time", the compact one to the "radius" x^0. Note that x^1 is light-like (typically the retarded time).

In conclusion, the quantity $\mathcal{K} \in \mathbb{Z}$ is a topological invariant, because it is completely independent of the metric and of the conformal frame. $\mathcal{K}/2$ can be interpreted as a winding number of a given path $\mathcal{P}$ around a reference point x. It is emphasized that this is *not* the case for all other nontrivial correlators of type (4.34). Indeed, $< X^a(x)e_a(y) >$ in eq. (4.35) receives similar contributions, but as its classical contribution from eq. (4.34) does not vanish the corresponding Wilson line is not a topological invariant, unless x coincides with a bifurcation point ($X^a = 0$).

The discussion above is in accordance with local quantum triviality but possible non-local quantum non-triviality after exact (background independent) quantum integration of the geometry.

4.4 Matter interactions

Although the result of the matterless case is an important consistency check of the calculations, the main motivation of this approach to quantum gravity is the study of matter interactions. Here the path integral after performing the momentum integrations of the matter fields takes the form

$$\mathcal{W}[j_{q_I}, j_p{}^I, J, J_\alpha] = \int \mathcal{D}(q^I, p_I, \mathfrak{q}, \mathfrak{q}^\alpha)(\det p_{++})^{1/2}\mathrm{sdet} M_I{}^J \exp\Big(i\int \mathrm{d}^2x\,\left(L_{(g)} + L_{(m)}\right)\Big)\,. \tag{4.43}$$

The determinant of p_{++} originates from the shifted $\mathfrak{p}$ integration, $L_{(g)}$ is the lagrangian (4.13) and $L_{(m)}$ the $\mathfrak{p}$ independent part of (4.12) supplemented by the source terms in J and J^α.

We denote the right hand side of the differential equations (4.14)–(4.18) by $q^I_{(g)}$. Then the related set of differential equations in presence of matter coupling can be abbreviated as

$$\begin{aligned}
\dot{q}^\phi &= q^\phi_{(g)}\,, && (4.44)\\
\dot{q}^{++} &= q^{++}_{(g)} - i\dot{\mathfrak{q}}^2 + \frac{1}{\sqrt{2}}\dot{\mathfrak{q}}^+\mathfrak{q}^+\,, && (4.45)\\
\dot{q}^{--} &= q^{--}_{(g)} - \frac{1}{\sqrt{2}}\dot{\mathfrak{q}}^-\mathfrak{q}^-\,, && (4.46)\\
\dot{q}^+ &= q^+_{(g)} - 2i\dot{\mathfrak{q}}\mathfrak{q}^+\,, && (4.47)\\
\dot{q}^- &= q^-_{(g)}\,. && (4.48)
\end{aligned}$$

The solutions of (4.44)–(4.48) are denoted by $\hat{B}^I$ and the remaining path integral becomes

$$\begin{aligned}
\mathcal{W}[j_{q_I}, j_p{}^I, J, J_\alpha] = &\int \mathcal{D}(\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}^\alpha)\,\times && (4.49)\\
&\times\exp\Big(i\int \mathrm{d}^2x\;\dot{\mathfrak{q}}\partial\mathfrak{q} - \frac{i}{\sqrt{2}}\mathfrak{q}^+\partial\mathfrak{q}^+ + \hat{B}^I j_{q_I} + \mathfrak{q}J + \mathfrak{q}^\alpha J_\alpha + \tilde{L}(j_p{}^I, \hat{B}^I)\Big)\,,
\end{aligned}$$

with $\mathcal{D}(\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}^\alpha)$ being the properly defined path integral measure with convenient superconformal properties [43, 44]. Its derivation is outlined in section 6. The term $\tilde{L}$ is produced in the same way as $\tilde{L}^0$ in (4.27),(4.28) for the matterless case. Because for all physical Green functions the sources of the target space coordinates are zero ($j_{q_I} = 0$) it actually encodes all nontrivial interactions, and in general is non-polynomial and nonlocal in the matter fields. Therefore, the remaining integrations over them cannot be performed exactly and one has to rely on a perturbation expansion.

5. Lowest order tree graphs

5.1 Localized matter

To simplify the derivation of the lowest order vertices[15] the concept of "localized matter" is useful [8, 9, 45]. To this end we introduce for the localized matter at some point y the notation

$$\Phi_0(x) = \frac{1}{2}\dot{\mathfrak{q}}^2(x) \Rightarrow a_0\delta^2(x-y)\,, \qquad \Phi_1(x) = \frac{1}{2}\partial\mathfrak{q}(x)\dot{\mathfrak{q}}(x) \Rightarrow a_1\delta^2(x-y)\,, \tag{5.1}$$

$$\Psi_0^{\pm\pm}(x) = \frac{1}{2}\dot{\mathfrak{q}}^{\pm}(x)\mathfrak{q}^{\pm}(x) \Rightarrow b_0^{\pm\pm}\delta^2(x-y)\,, \qquad \Psi_1^{\pm\pm}(x) = \frac{1}{2}\partial\mathfrak{q}^{\pm}\mathfrak{q}^{\pm} \Rightarrow b_1^{\pm\pm}\delta^2(x-y)\,, \tag{5.2}$$

$$\Pi_0^{\pm}(x) = \dot{\mathfrak{q}}(x)\mathfrak{q}^{\pm}(x) \Rightarrow c_0^{\pm}\delta^2(x-y)\,. \tag{5.3}$$

From the matter interaction terms in the gauge-fixed lagrangian (4.43)

$$L_{\text{int}} = 2i\Phi_0 p_{++} - \sqrt{2}\Psi_0^{++}p_{++} + \sqrt{2}\Psi_0^{--}p_{--} + 2i\Pi_0^{+}p_{+}\,, \tag{5.4}$$

the lowest order vertices can be determined by solving to first order in localized matter (5.1)–(5.3) the *classical* equations of motion of the geometrical variables involved. To this end the asymptotic integration constants must be chosen in a convenient way. Following the calculations of the purely bosonic case [8, 10] $q^{\phi}(x^0\to\infty) = x^0$, which implies $q^{++}(x^0\to\infty) = i$. In addition $p_{--}(x^0\to\infty) = ie^{Q}$ may now be imposed. Finally we have to fix the asymptotic value of the Casimir function (2.11): $C(x^0\to\infty) = C_\infty$. Due to the matter interactions $\mathrm{d}C \neq 0$, but the conservation law receives contributions from the matter fields as well [26]. The exact form of the fermionic integration constants depends on C_∞. If $C_\infty = 0$ there exists (asymptotically) a second Casimir function $\tilde{c}$ [23, 25]. Then the value of $\tilde{c}$ determines one asymptotic integration constant from the fermions.

The equations of motion of the dilaton, the dilatino and q^{++} can be solved explicitly for general asymptotic values q^+_∞ and q^-_∞. However, the dependence of q^{--} on the different target-space variables q^I is more complicated (cf. (4.16)) and the fermion dependent part (soul) of that differential equation cannot be integrated for general functions Q and W. In the current work this restriction does not have far-reaching consequences: In any scattering problem the most natural and technically manageable boundary conditions have asymptotically vanishing values of the fermion fields and thus all fermionic integration constants are set to zero: $q^+_\infty = q^-_\infty = 0$. Then the Casimir function has no soul while its body becomes

$$C = -m_\infty + \Big[i(2ia_0 - \sqrt{2}b_0^{++})\Big(m_\infty + \big[W\big]_{y^0}\Big) + \sqrt{2}ie^{Q}b_0^{--}\Big]h\,. \tag{5.5}$$

[15] Alternatively one can derive them by brute force methods encountering the difficulty how to define properly operators of the type ∂_0^{-1}. Although this can be done [8] the calculations become rather lengthy already in the bosonic case. Therefore, only the simpler method described in the text will be presented, but it is emphasized that of course both methods give the same answer. For higher order $(2n+2)$-point vertices matter will have to be localized at n different points, but here only the case $n = 1$ will be studied.

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Square brackets are used to emphasize the terms which depend on y^0. The new abbreviation $h = \theta(y^0 - x^0)\delta(x^1 - y^1)$ has been used in this equation. m_∞ is the integration constant of $q^{--} = ie^{-Q}m_\infty + \cdots$, which, however, turns out to be equivalent to the asymptotic value $-C_\infty$. Notice that all contributions except this integration constant are proportional to h and thus all geometric variables may be replaced by their asymptotic values in that equation. Of course, this is equivalent to the statement that C is constant in the absence of matter fields.

5.2 Four point vertices

To simplify the solution of the equations of motion for p_I, which determine the vertices according to (5.4), we select the special asymptotic geometry with $C_\infty = 0$. This implies [26] the BPS condition[16] and for Minkowski ground-state models space-time becomes asymptotically flat. To first order in localized matter the relevant equations of motion can be written as (cf. [26]):

$$\partial_0 p_{--} = -iq^{++}Zp_{--}\,, \tag{5.6}$$

$$\partial_0 p_{++} = ip_\phi - iq^{++}q^{--}e^Q Z\,, \tag{5.7}$$

$$\partial_0 p_\phi = -ie^Q\big((e^{-Q}W')' + q^{++}q^{--}Z'\big)\,, \tag{5.8}$$

$$\partial_0 p_+ = \frac{i}{2\sqrt{2}}q^- e^{-Q/2}(\sqrt{-W})''p_{--} + ie^{-Q/2}(\sqrt{-W})'p_-\,, \tag{5.9}$$

$$\partial_0 p_- = \frac{-i}{2\sqrt{2}}q^+ e^{-Q/2}(\sqrt{-W})''p_{--} - i\frac{Z}{2}q^{++}p_-\,. \tag{5.10}$$

Notice that all terms quadratic in the fermions in (5.7) and (5.8) vanish as they are second order in localized matter. The solution of (5.6) is simply $p_{--} = ie^Q$ and the expansion to first order in the matter fields yields

$$p_{--} = ie^Q\big(1 - iZ(2ia_0 - \sqrt{2}b_0^{++})(x^0 - y^0)h\big)\,. \tag{5.11}$$

Using the helpful relation $\partial_0 = -iq^{++}\partial_\phi$ the equation of motion for p_+ can be integrated easily. For later convenience it is abbreviated as

$$p_+ = -\frac{1}{\sqrt{2}}c_0^+ hV_3(x,y)\,, \tag{5.12}$$

$$V_3(x,y) = \Big((\sqrt{-W})' - \Big[(\sqrt{-W})'\Big]_{y^0}\Big)\Big((\sqrt{-W}) - \Big[(\sqrt{-W})\Big]_{y^0}\Big)\,. \tag{5.13}$$

As the whole expression is proportional to c_0^+ all ϕ dependent quantities simply depend on x^0.

[16] Clearly the asymptotic fields cannot obey both supersymmetries. This would imply that $q^{\pm\pm}$ and $p_{\pm\pm}$ vanish asymptotically and thus the (bosonic) line element would be degenerate in this limit.

The equation of motion for p_{++} reads

$$(q^{++}p'_{++})' = -W''\big(1 + i(2ia_0 - \sqrt{2}b_0^{++})\big) + \sqrt{2}ib_0^{--}(e^Q h)'Z\,. \tag{5.14}$$

This yields two analytically different types of contributions to the vertices and thus the solution is written as

$$p_{++} = iW(x^0) + (2ia_0 - \sqrt{2}b_0^{++})hV_1(x,y) + \sqrt{2}b_0^{--}\big(\left[e^Q Z\right]_{y^0}(x^0 - y^0)h + V_2(x,y)\big)\,. \tag{5.15}$$

The differential equation can be solved explicitly for V_1

$$V_1 = \big(W' + \left[W'\right]_{y^0}\big)(x^0 - y^0) - 2\Big(W - \left[W\right]_{y^0}\Big)\,, \tag{5.16}$$

while V_2 in general cannot be integrated:

$$\partial_0^2 V_2 = e^Q Z^2 h\,. \tag{5.17}$$

It is important to realize that the two integration constants hidden in V_2 are determined by the asymptotic value ($x^0 \to \infty$) of p_{++} in (5.15). Because for this value our boundary conditions yield $iW(x^0)$ both integration constants have to vanish for $x^0 > y^0$, so there is no ambiguity from homogeneous solutions to (5.17) anymore. Exploiting that to leading order $\dot{f} := df/dx^0 = df/d\phi$ the differential equation (5.17) can be integrated in the regions $x^0 > y^0$, yielding $V_2^> = 0$, and in the region $x^0 < y^0$

$$V_2^< = \tilde{A} + \tilde{B}x^0 + e^{Q(x^0)} + \int^{x^0}\int^{x^{0\prime}} e^{Q(x^{0\prime\prime})}\dot{Z}(x^{0\prime\prime})dx^{0\prime\prime}dx^{0\prime}\,. \tag{5.18}$$

The matching conditions are given by continuity of V_2 and its first derivative at $x^0 = y^0$ (note that this is not a requirement imposed arbitrarily but a consequence from the θ-function hidden in h):

$$0 = \tilde{A} + \tilde{B}y^0 + e^{Q(y^0)} - \int^{y^0}\int^{y'} e^{Q(y'')}\dot{Z}(y'')dy''dy'\,, \tag{5.19}$$

$$0 = \tilde{B} + e^{Q(y^0)}Z(y^0) - \int^{y^0} e^{Q(y')}\dot{Z}(y')dy'\,. \tag{5.20}$$

Thus, the full vertex V_2 is given by $V_2 = V_2^< h$. As a nontrivial example we perform the integrations for the physically relevant class of models[17] with $Z = -a/\phi$

$$V_2|_{Z=-a/\phi} = \frac{a}{a+1}\Big((x^0)^{-a} - (y^0)^{-a} + a(y^0)^{-(a+1)}(x^0 - y^0)\Big)\,h\,. \tag{5.21}$$

The different vertices $\mathcal{V}$ now follow straightforwardly. V_1 gives rise to a term of the type $\dot{\mathfrak{q}}\dot{\mathfrak{q}}(x^0) \to \dot{\mathfrak{q}}\dot{\mathfrak{q}}(y^0)$ from the first, $\dot{\mathfrak{q}}^+\mathfrak{q}(x^0) \to \dot{\mathfrak{q}}^+\mathfrak{q}(y^0)$ from the second and the mixed vertex from the first two terms in (5.4):

$$\mathcal{V}\big(\dot{\mathfrak{q}}\dot{\mathfrak{q}}(x^0) \to \dot{\mathfrak{q}}\dot{\mathfrak{q}}(y^0)\big) \;=\; -4V_1(x,y)h\,, \tag{5.22}$$

[17]E.g. in the bosonic part of spherically reduced Einstein gravity $a = (D-3)/(D-2)$. In the CGHS model $a = 1$ [15].

$$\mathcal{V}\big(\dot{\mathfrak{q}}^+\mathfrak{q}^+(x^0) \to \dot{\mathfrak{q}}^+\mathfrak{q}^+(y^0)\big) \;=\; 2V_1(x,y)h\,, \tag{5.23}$$

$$\mathcal{V}\big(\dot{\mathfrak{q}}\dot{\mathfrak{q}}(x^0) \to \dot{\mathfrak{q}}^+\mathfrak{q}^+(y^0)\big) \;=\; -2\sqrt{2}iV_1(x,y)h\,. \tag{5.24}$$

Considering the last expression it is important to notice that the two different contributions from (5.4) add, as $h(x,y)V_1(x,y)$ is symmetric in x and y.

In a similar way the vertices from V_2 follow. Here the contributions from the third term in (5.4) and the explicitly integrated part $[e^Q Z]_{y^0}$ cancel and one obtains

$$\mathcal{V}\big(\dot{\mathfrak{q}}\dot{\mathfrak{q}}(x^0) \to \dot{\mathfrak{q}}^-\mathfrak{q}^-(y^0)\big) \;=\; 2\sqrt{2}iV_2(x,y)\,, \tag{5.25}$$

$$\mathcal{V}\big(\dot{\mathfrak{q}}^+\mathfrak{q}^+(x^0) \to \dot{\mathfrak{q}}^-\mathfrak{q}^-(y^0)\big) \;=\; -2V_2(x,y)\,. \tag{5.26}$$

Finally, V_3 yields vertices with mixed initial and final states:

$$\mathcal{V}\big(\dot{\mathfrak{q}}\mathfrak{q}^+(x^0) \to \dot{\mathfrak{q}}\mathfrak{q}^+(y^0)\big) = \frac{-i}{\sqrt{2}}V_3(x,y)h\,. \tag{5.27}$$

Note that the set of vertices is invariant under the exchange $\dot{\mathfrak{q}}\dot{\mathfrak{q}} \leftrightarrow i\dot{\mathfrak{q}}^+\mathfrak{q}^+/\sqrt{2}$. Moreover, the vertices V_1 and V_3 are conformally invariant, but V_2 is not. In addition, it should be pointed out that all vertices vanish in the coincidence limit $x^0 = y^0$. This is of relevance for the elimination of nonlocal loops: it seems likely that the arguments in favor of such a cancellation presented in appendix B.2 of ref. [41] can be extended to the present case (to visualize a "nonlocal loop" we refer to the graphs 2,3,4 and 6 in figure B.6 of that reference).

Another remark concerns the special cases where one or more of the vertices vanish. Clearly, for non-dynamical dilaton, $Z = 0$, no contribution arises from the non-invariant vertex V_2. It is worthwhile mentioning that V_2 is independent of W, the "good" function in the parlance of [41] and solely depends on the "muggy" one. In contrast, the invariant vertices solely depend on W — which, indeed, is the very reason for their invariance.

5.3 Implications for the S-matrix

The vertices derived in the previous section are the first step to derive S-matrix elements. However, they are the most important one, as from now on the calculation is rather straightforward: one has to introduce asymptotic states, which are very simple for vanishing C_∞, build a corresponding Fock space, insert the asymptotic states in the vertices (5.22)–(5.27) and perform all the integrations involved. Unfortunately, this method already in the bosonic case [45] requires a lot of efforts (see appendix F of [27] for details). Thus, we will restrict ourselves to some of the general features that can be discussed without actually performing all these steps.

For $p_+ = 0$ at the boundary one obtains for the asymptotic states

$$\partial_0\left(\partial_1\mathfrak{q} - E_1^{--}\partial_0\mathfrak{q}\right) = 0\,, \tag{5.28}$$

$$\partial_1\mathfrak{q}^+ - \partial_0\left(E_1^{--}\mathfrak{q}^+\right) = 0\,, \tag{5.29}$$

$$\partial_0\left(E_1^{++}\mathfrak{q}^-\right) = 0\,, \tag{5.30}$$

with $E_1^{--} = -ip_{++}|_{as} = W|_{as}$ and $E_1^{++} = -ip_{--}|_{as} = e^Q|_{as}$. The first two of these equations are conformally invariant because in the present gauge E_1^{--} is invariant while

E_1^{++} is not. This matches nicely with the conformal invariance of the vertices V_1 and V_3 derived in the previous subsection. Thus, contributions from (5.22)–(5.24) and from (5.27) to the S-matrix are also conformally invariant. The last equation above implies that $\mathfrak{q}^-$ is not conformally invariant. However, neither is the vertex V_2. Thus, it is conceivable that together with the non-invariance of the asymptotic states the noninvariant vertices V_2 also yield an invariant contribution to the S-matrix in (5.25)–(5.26).

Actually, we will show now that the contributions from V_2 with external legs to the S-matrix are not only conformally invariant but they *vanish identically*: By definition one of the external legs on the left hand side of (5.25) and (5.26) consists of $\dot{\mathfrak{q}}^-\mathfrak{q}^-$. But since the solution of (5.30) contains a fermionic integration constant $\mathfrak{q}_0^-(x^1)$ and the latter appears quadratically in $\dot{\mathfrak{q}}^-\mathfrak{q}^-$ this term vanishes identically. Similarly, it can be argued that (5.23), (5.24) yield no contribution to the S-matrix. *Thus, for the lowest order tree-level S-matrix one has to take into account only (5.22) and (5.27).*[18]

For constant W (so-called "generalized teleparallel" theories, including rigid supersymmetry with cosmological constant) these vertices vanish identically. Less trivial is the special case $W \propto \phi$ (this family of models has been studied in the bosonic case by Fabbri and Russo [46] and it includes the CGHS model [14]): the vertex V_1 vanishes, but V_3 does not. Thus, although the CGHS model exhibits the feature of scattering triviality [10], its supersymmetric version loses this property. If $W \propto \phi^2$ the vertex V_3 does not contribute. This happens, for instance, for supergravity extensions of the Jackiw-Teitelboim model [16]. In this case the lowest order supergravity scattering amplitude is equivalent to its bosonic counter part.

6. Measure and 1-loop action

In eq. (4.49) the proper definition of the remaining path integral measure over the matter fields remained open, which is needed in calculations of matter loops. We follow the steps performed already in the purely bosonic case [15, 11]: as a suitable choice of the measure is known for the quantization of the matter fields on a fixed background [43], the path integral in (4.49) is reduced to that problem. Nevertheless, the situation is more complicated in the present application: The result for the supersymmetric path integral measure [44] is derived from a linear realization of supersymmetry while the gPSM based version deals with a non-linear one. Thus, we suggest to extend the approach of [11] in such a way that the results of ref. [44] should remain applicable to our case.

The path integral measure of the linear theory may be written as $\mathcal{D}\tilde{M} = \mathcal{D}(E^{1/2}M)$, where $M = f - i\theta\lambda + \frac{1}{2}\theta^2 H$ is the matter multiplet in superspace and E the superdeterminant. The covariant fields $\tilde{f}$, $\tilde{\lambda}_\alpha$ and $\tilde{H}$ depend on the vielbein components e_m^a, the (superspace) gravitino[19] Ψ_m^α and the auxiliary fields of the matter sector A. Thus auxiliary background fields $\check{e}^a$, $\check{\Psi}^\alpha$ and $\check{A}$ are introduced, their "on-shell" values are determined by

[18]Nevertheless, the other vertices will be of relevance for calculations of higher order in loops and matter, which is why we presented them in their full glory.

[19]The gravitino in the MFS model ψ_m^α is not identical with the superspace field Ψ_m^α, but related by $\Psi_m^\alpha = \psi_m^\alpha - \frac{i}{8} Z e_m^a \epsilon_{ab} (\chi\gamma^b)^\alpha$ [25].

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means of the operators

$$\hat{e}_1^{\pm\pm} = -i\frac{\delta}{\delta {j_p}^{\mp\mp}}\,, \qquad \hat{e}_0^{--} = -i\,, \qquad \hat{e}_0^{++} = 0\,, \tag{6.1}$$

$$\hat{A} = -\frac{1}{2}(u' + uZ) - \frac{1}{8}Z'\frac{\delta}{\delta j_{q_-}}\frac{\delta}{\delta j_{q_+}}\,, \tag{6.2}$$

$$\hat{\Psi}_1^{\pm} = \mp i\frac{\delta}{\delta {j_p}^{\mp}} - \frac{1}{4\sqrt{2}}Z\frac{\delta}{\delta {j_p}^{\mp\mp}}\frac{\delta}{\delta j_{q_\mp}}\,, \qquad \hat{\Psi}_0^{-} = -\frac{1}{4\sqrt{2}}Z\frac{\delta}{\delta j_{q_+}}\,, \qquad \hat{\Psi}_0^{+} = 0\,, \tag{6.3}$$

where in all dilaton-dependent functions the replacement $q^\phi \to -i\frac{\delta}{\delta j_{q_\phi}}$ must be made. The path integral is rewritten as

$$\mathcal{W}[j_{q_I}, {j_p}^I, J, J_\alpha] = \int \mathcal{D}(\check{e}_m^a, \check{\Psi}_m^\alpha, \check{A})\delta(\check{e}_m^a - \hat{e}_m^a)\delta(\check{\Psi}_m^\alpha - \hat{\Psi}_m^\alpha)\delta(\check{A} - \hat{A})\tilde{\mathcal{W}}\,. \tag{6.4}$$

In $\tilde{\mathcal{W}}$ all geometrical variables in the measure may be replaced by auxiliary variables and thus the integration reduces to a quantization on a fixed background. In our gauge the necessary redefinitions read:[20]

$$\tilde{f} = \sqrt{\check{e}}f\,, \qquad \tilde{\lambda}^\alpha = \sqrt{\check{e}}\lambda^\alpha + \frac{1}{2}\sqrt{\check{e}}f(\check{\Psi}^a\gamma_a)^\alpha\,, \qquad \tilde{H} = \sqrt{\check{e}}H + \check{\Psi}\tilde{\lambda} + \frac{1}{2}\tilde{f}\left(\check{A} - \frac{1}{\check{e}}\check{\Psi}_0^-\check{\Psi}_1^+\right). \tag{6.5}$$

Taking into account the ambiguous terms from (4.49) $\tilde{\mathcal{W}}$ in this gauge may be written as

$$\begin{aligned}\tilde{\mathcal{W}}[\check{e}_m^a, \check{\Psi}_m^\alpha, \check{A}, {j_p}^I, j_{q_I}, J, J_\alpha] = \int \mathcal{D}(\tilde{f}, \tilde{\lambda}_\alpha, \tilde{H}) \times \\ \times \exp\Bigg(i\int \mathrm{d}^2x\,\big(i\check{e}_{++}\dot{f}^2 + \dot{f}(\partial f - 2i\check{\Psi}_+\lambda^+) - \\ - \frac{1}{\sqrt{2}}\check{e}_{++}\dot{\lambda}^+\lambda^+ + \frac{1}{\sqrt{2}}\check{e}_{--}\dot{\lambda}^-\lambda^- - \frac{i}{\sqrt{2}}\lambda^+\partial\lambda^+ - \\ - \frac{i}{2}\check{e}_{--}H^2 + fJ + \lambda^\alpha J_\alpha + \Delta L\big)\Bigg)\,.\end{aligned} \tag{6.6}$$

Here, ΔL represents all terms, which do not contribute to the quadratic part in the matter fields. To make contact with the calculations from the linear theory, an additional gaussian factor $\check{e}H^2$ has been introduced (which in the present gauge becomes $\check{e}_{--}H^2$).

After the reformulation of the matter loop expansion as quantization on a fixed background, existing results from the literature may be taken over.[21] We do not present any formulae in component expansion, but the more convenient superspace expressions. Of course, in any concrete applications the component expansion will become manifest. However it is expected that many terms thereof will be irrelevant for the leading quantum corrections.

[20] As the calculation of this section are performed at the lagrangian level, canonical coordinates $\mathfrak{q}$, $\mathfrak{q}^\pm$ are again written as f, $\lambda^\pm$.

[21] Though our formulation of supergravity deals with a non-linear realization thereof the matter action (2.24) is — up to the gaussian factor H^2 — equivalent to the one obtained from superspace. Thus no additional complications with auxiliary fields arise. However, it should be mentioned that our argument relies on an important assumption: The results obtained from the regularized linear theory can be used directly in our formalism after having integrated out all geometric variables.

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After a careful implementation of the source terms, the supersymmetric extension of the Polyakov action [47] in a gauge-independent description[22] becomes [48]

$$\mathcal{S}_{\text{eff}} = \int_X \int_Y \left(\frac{1}{24\pi} (ES)(X) \Delta(X,Y) S(Y) - \frac{1}{2} \mathcal{J}(X) \Delta(X,Y) \mathcal{J}(Y) \right) . \tag{6.7}$$

Here $X = (x, \theta)$ and $Y = (y, \tilde{\theta})$ are two sets of superspace coordinates and Δ the Green function of the quadratic superspace derivative $\frac{1}{2} D^\alpha D_\alpha \Delta = \delta^4(X - Y)$.

7. Conclusions

A background independent non-perturbative quantization of two dimensional dilaton supergravity has been presented. The relevant steps rely on the first order formulation [25] of superfield supergravity [2] in terms of a graded Poisson-sigma model [4], and a hamiltonian path integral quantization. Remarkably, no quartic ghost terms arise.

In the matterless case *all* integrations can be done exactly. The ensuing effective action is — up to boundary terms — equivalent to the gauge-fixed classical action, which confirms the local quantum triviality of the model as expected from the bosonic case [6, 7, 8]. Nevertheless, there exist interesting non-local quantum correlators. For a specific class thereof a topological interpretation of the non-perturbative quantum effects exists. It could be of interest to apply these results to super Liouville theory (cf. e.g. [49]).

With matter couplings the integration over all geometrical variables still can be performed exactly, but the matter interactions have to be treated perturbatively. For this case the (non-local) four-point vertices have been determined to lowest order in matter. As in the bosonic theory the phenomenon of virtual black holes [9, 10] is encountered, but supergravity yields a richer structure of vertices. In particular the supersymmetric CGHS model — in contrast to its bosonic counterpart — does not exhibit scattering triviality at tree level. Finally it has been argued that matter loop corrections should be obtainable from quantization on a fixed background. In particular, the one loop results follow from the super-Polyakov action.

A natural next step will be the calculation of S-matrix elements for simple scenarios by virtue of the vertices (5.22)-(5.24) and (5.27), in analogy to the bosonic case [45]. A different aspect to be investigated in detail are matter loops. At least for specific (simple) models explicit computations of corrections (analogous to the results of ref. [11] in the bosonic theory) should be feasible.

So far, all results on matter interactions are restricted to the minimally coupled, conformal case. One might try to extend the quantization procedure to non-minimally coupled fields and/or self-interactions. However, additional technical difficulties arise in these cases. Indeed, an important simplification in the calculations of sections 3 and 4 has been the fact that $G^{++}_{(m)}$ is independent of p_{--}, $G^{+}_{(m)}$ independent of p_{-} etc. These restrictions are lost together with non-minimal coupling or self-interaction, which makes the calculation of

[22]The superspace conventions are taken from [28, 25]. $\mathcal{J} = J_H + i\theta^\alpha J_\alpha + \frac{1}{2}\theta^2 J$ is the superfield of sources, S the supergravity multiplet and E the superdeterminant of ref. [1].

the constraint algebra much more complicated (cf. the difficulties that already arose in the purely bosonic case [27, 15]). Also it is not obvious that the homological perturbation theory still stops at Yang-Mills level. If this were no longer the case, an important ingredient justifying the approach outlined in this work may be lost. Nevertheless, preliminary calculations lead to promising results which suggest that some generalizations can be treated by the program outlined in this work.

Finally, two of the present authors have shown recently [50] that the gPSM approach applies to models with extended supergravity as well, which suggests further interesting generalizations of the current work.

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A. Notations and conventions

These conventions are identical to [23, 28], where additional explanations can be found.

Indices chosen from the Latin alphabet are commuting (lower case) or generic (upper case), Greek indices are anti-commuting. Holonomic coordinates are labeled by M, N, O etc., anholonomic ones by A, B, C etc., whereas I, J, K etc. are general indices of the gPSM. The index ϕ is used to indicate the dilaton component of the gPSM fields:

$$X^{\phi} = \phi \qquad A_{\phi} = \omega \,. \tag{A.1}$$

The summation convention is always $NW \to SE$, e.g. for a fermion χ: $\chi^2 = \chi^{\alpha}\chi_{\alpha}$. Our conventions are arranged in such a way that almost every bosonic expression is transformed trivially to the graded case when using this summation convention and replacing commuting indices by general ones. This is possible together with exterior derivatives acting *from the right*, only. Thus the graded Leibniz rule is given by

$$\mathrm{d}\,(AB) = A\mathrm{d}B + (-1)^{B}\,(\mathrm{d}A)B \,. \tag{A.2}$$

In terms of anholonomic indices the metric and the symplectic 2×2 tensor are defined as

$$\eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \epsilon_{ab} = -\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.3}$$

The metric in terms of holonomic indices is obtained by $g_{mn} = e_n^b e_m^a \eta_{ab}$ and for the determinant the standard expression $e = \det e_m^a = \sqrt{-\det g_{mn}}$ is used. The volume form reads $\epsilon = \frac{1}{2}\epsilon^{ab} e_b \wedge e_a$; by definition $*\epsilon = 1$.

The γ-matrices are used in a chiral representation:

$$\gamma^0{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_{*\alpha}{}^\beta = (\gamma^1\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

Covariant derivatives of anholonomic indices with respect to the geometric variables $e_a = \mathrm{d}x^m e_{am}$ and $\psi_\alpha = \mathrm{d}x^m \psi_{\alpha m}$ include the two-dimensional spin-connection one form $\omega^{ab} = \omega\epsilon^{ab}$. When acting on lower indices the explicit expressions read ($\frac{1}{2}\gamma_*$ is the generator of Lorentz transformations in spinor space):

$$(De)_a = \mathrm{d}e_a + \omega\epsilon_a{}^b e_b\,, \qquad (D\psi)_\alpha = \mathrm{d}\psi_\alpha - \frac{1}{2}\omega\gamma_{*\alpha}{}^\beta\psi_\beta\,. \tag{A.5}$$

Light-cone components are very convenient. As we work with spinors in a chiral representation we can use

$$\chi^\alpha = (\chi^+, \chi^-)\,, \qquad \chi_\alpha = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}. \tag{A.6}$$

For Majorana spinors upper and lower chiral components are related by $\chi^+ = \chi_-$, $\chi^- = -\chi_+$, $\chi^2 = \chi^\alpha\chi_\alpha = 2\chi_-\chi_+$. Vectors in light-cone coordinates are given by

$$v^{++} = \frac{i}{\sqrt{2}}(v^0 + v^1)\,, \qquad v^{--} = \frac{-i}{\sqrt{2}}(v^0 - v^1)\,. \tag{A.7}$$

The additional factor i in (A.7) permits a direct identification of the light-cone components with the components of the spin-tensor $v^{\alpha\beta} = \frac{i}{\sqrt{2}}v^c\gamma_c^{\alpha\beta}$. This implies that $\eta_{++|--} = 1$ and $\epsilon_{--|++} = -\epsilon_{++|--} = 1$. The γ-matrices in light-cone coordinates become

$$(\gamma^{++})_\alpha{}^\beta = \sqrt{2}i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad (\gamma^{--})_\alpha{}^\beta = -\sqrt{2}i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A.8}$$

B. Details relevant for the path integral

B.1 Boundary terms

The action (2.1) may consistently be extended by a boundary term

$$\mathcal{S}_{gPSM} = \int_M \mathrm{d}X^I \wedge A_I + \frac{1}{2}P^{IJ}A_J \wedge A_I - \int_{\partial M} f(C)X^I A_I \tag{B.1}$$

with $f(C)$ an arbitrary function depending on the Casimir function. Beside the bulk field equations (2.5) and (2.6) this yields the following conditions on the boundary

$$f(C)X^I\delta A_I\big|_{\partial M} = 0\,, \tag{B.2}$$

$$\delta X^I\left[(f(C)-1)A_I + \partial_I C f'(C)X^J A_J\right]\big|_{\partial M} = 0\,. \tag{B.3}$$

B.1.1 Simplest boundary conditions

A consistent[23] way is fixing $f(C)=0$ and $\delta X^I=0$ at the boundary, which for a time-like boundary means $X^I\big|_{\partial M}=X^I(r)$, where in the gauge chosen in this work $r=r(x^0)$. For the time being we will assume this boundary prescription, which has been used e.g. in the quantization of spherically reduced gravity [8, 9].

With respect to the symmetry variation (2.3) the action (B.1) with $f(C)=0$ transforms into a surface integral

$$\delta_\varepsilon \mathcal{S}_{gPSM}=\int_{\partial M}\mathrm{d}X^I\varepsilon_I\,. \tag{B.4}$$

For the current boundary prescription this term vanishes since X^I are fixed on the (time-like) boundary. Note that also further variations and/or symmetry transformations of (B.4) vanish, since ε depends only on the world-sheet coordinates x and the fields X, and the symmetry variation of X again yields a function of X.

The commutator of two symmetry variations

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\,X^I\;=\;\delta_{\varepsilon_3}X^I\,, \tag{B.5}$$

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\,A_I\;=\;\delta_{\varepsilon_3}A_I+\left(\mathrm{d}X^J+P^{JK}A_K\right)\partial_J\partial_I P^{RS}\varepsilon_{1\,S}\varepsilon_{2\,R}\,, \tag{B.6}$$

in general only closes on-shell, with

$$\varepsilon_{3\,I}=\partial_I P^{JK}\varepsilon_{1\,K}\varepsilon_{2\,J}+P^{JK}\left(\varepsilon_{1\,K}\partial_J\varepsilon_{2\,I}-\varepsilon_{2\,K}\partial_J\varepsilon_{1\,I}\right). \tag{B.7}$$

By applying two consecutive symmetry variations (in order to get insight into trivial large gauge transformations) we obtain

$$\delta_{\varepsilon_1}\delta_{\varepsilon_2}L=\int_{\partial M}\left(\mathrm{d}X^I P^{JK}\varepsilon_{1\,K}\partial_J\varepsilon_{2\,I}-P^{IJ}\varepsilon_{1\,J}\,\mathrm{d}\varepsilon_{2\,I}\right). \tag{B.8}$$

For fixed X^I at the boundary the first term vanishes, while the second one can yield only something in the direction orthogonal to $\mathrm{d}X^K$. e.g. in purely bosonic first order gravity the boundary values were fixed to $X^k=X^k(r)$ and thus (in Schwarzschild coordinates) only a dt component can survive in (B.8). Thus, the boundary action is of the form $\int dtM\dot{p}_M$. This is essentially the result that Kuchar obtained for the Schwarzschild black hole [52]. It is also very similar to what Gegenberg, Kunstatter and Strobl obtained in Casimir-Darboux coordinates (cf. eqs. (38-42) in [53]). The application of further symmetry transformations does not change the structure of this result.

B.1.2 Other boundary conditions

Now we discuss other possible boundary prescriptions, always assuming that on the boundary $A_I\neq 0\neq X^I$ in general (because e.g. in gravity natural boundary conditions would be most "unnatural" since the metric would degenerate). Instead of $\delta X^I=0$ we can require

[23] We mean consistency as defined in [51]: boundary conditions arise from (1) extremizing the action, (2) invariance of the action under symmetry transformation and (3) closure of the set of boundary conditions under symmetry transformations.

$\delta A_I = 0$, automatically fulfilling (B.2). We now want to avoid $\delta X^I = 0$ (which would obey (B.3) trivially) and therefore must require

$$\left[(f(C)-1)A_I + \partial_I C f'(C) X^J A_J\right]\Big|_{\partial_M} = 0\,. \tag{B.9}$$

For consistency, also $\delta_\varepsilon \delta A_I = 0$ on the boundary. This yields the condition

$$\partial_L(\partial_I P^{JK})\varepsilon_K = 0\,, \tag{B.10}$$

which is satisfied for a linear Poisson tensor (i.e. in the Yang-Mills case), since in that case also $\varepsilon = \varepsilon(x)$. Then the well-known relations for these models are reproduced (cf. e.g. [54] for the abelian case). For more general theories eq. (B.10) would restrict our symmetry transformations asymptotically, which we excluded in this work. Other possible boundary prescriptions are either $\delta A_I = A_I = 0$ or $\delta X^I = X^I = 0$ at the boundary with arbitrary $f(C)$. But as mentioned before neither of them can be used in (super)gravity.

Since a symmetry transformation mixes the components of X^I a consistent set of mixed boundary prescriptions does not exist in general. e.g. the requirements $\delta X^1 = 0$, $A_2 = \cdots = A_J = 0$ and $f = 0$ fulfill (B.2) and (B.3), but the symmetry variation of X^1 yields $P^{1J}\varepsilon_J$, which in general will depend on all X^I, and a variation thereof need not vanish.

In summary we conclude that for the most general gPSM only the boundary prescription $\delta X^I = 0$ and $f(C) = 0$ is consistent, while for certain special cases (essentially Yang-Mills) alternative prescriptions are possible.

One might wonder about consistent boundary conditions for the matter fields as well. Certainly, this is an interesting question when considering global objects (solitons). However, the quantization presented in this work treats the geometrical variables non-perturbatively while the matter fields only can be included in a perturbative framework. In that case all matter fields can be assumed to fulfill natural boundary conditions.

B.2 Ordering

Generic gPSM gravity (with respect to the Poisson bracket (3.5)) as well as MFS minimally coupled to matter (with respect to the Dirac bracket (3.21)) is free of ordering problems if we require hermiticity[24] of the hamiltonian. To this end we show the validity of three statements:

1. Any hermitian operator version of the classical hamiltonian is automatically Weyl ordered.

   As the hamiltonian is a sum over the constraints (3.14), while the latter are independent of $\bar{p}_I$, it is Weyl ordered if the constraints have that property. For the geometrical part the statement follows from their linearity in p_I. E.g. the $G^I_{(g)}$ can be written in a hermitian version as

$$G^I_{(g)} = \partial q^I + \frac{1}{2}\left(P^{IJ}(q)p_J + (-1)^{J(I+1)} p_J P^{IJ}(q)\right), \tag{B.11}$$

[24] Although it is not indicated explicitly, all formulas in this subsection refer to operator expressions.

which is Weyl ordered since every commutator with q from the right is compensated by another commutator from the left. Thus in that part of the hamiltonian no ordering terms can appear.

The situation in the matter hamiltonian is almost trivial. As this part of H is independent of the target space coordinates q^I, the Dirac bracket does not lead to complications (cf. (3.24)–(3.28)). The matter fields do appear at most quadratically and thus Weyl ordering is trivial.

2. The commutator $[G^K, C_K{}^{IJ}]$ vanishes even at the quantum level.

   For the gPSM part one simply notices that this commutator (for any ordering prescription, even non-hermitian ones) is given by

$$[G^K_{(g)}, C_K{}^{IJ}] = -P^{KL}\partial_L\partial_K P^{IJ} = 0 \tag{B.12}$$

   due to (anti-)symmetry of the Poisson tensor.

   More involved is the discussion of $G^I_{(m)}$. Indeed, it seems that there could appear contributions to this commutator, whenever a $\mathfrak{q}^+$ or $1/p_{++}$ ($\mathfrak{q}^-$ or $1/p_{--}$) hits a q^{++} (q^{--}) from the structure functions (cf. (3.29)–(3.32)). Also the term $\propto p_\pm$ in $G^{\pm\pm}_{(m)}$ is not obviously seen to commute. But it turns out that the supergravity restrictions (2.14) and (2.15) resolve all problems:

   - All structure functions $C_{++}{}^{IJ}$ are independent of q^{++} as the Poisson tensor is linear in that variable.
   - All structure functions $C_{++}{}^{IJ}$ are independent of q^+ as well, which (for the non-trivial terms) is a consequence of (2.15).
   - All structure functions $C_+{}^{IJ}$ are independent of q^{++}, which can again be read off from (2.14) and (2.15) without use of the explicit solution (2.16)–(2.18). Notice that $\partial_{++}C_+{}^{++|--} = 0$ due to the first equation in (2.15): As $\partial_+ P^{+|--}$ is independent of χ^+, $\partial_+\partial_{++}P^{++|--} = 0$.

   Thus we have shown that the commutator $[G^K_{(m)}, C_K{}^{IJ}] = 0$. In fact each individual contribution from the l.h.s. of that relation vanishes separately.

3. The commutator of two Weyl ordered constraints yields the structure functions times the Weyl ordered constraints (or, equivalently, the Weyl ordered product of constraints times the structure functions).

   Using the result for $G^I_{(m)}$ from the second point above this statement is obvious for that part of the constraints. For $G^I_{(g)}$ we can use their form (B.11). The commutator between G^I and G^J yields several terms, including ordering terms proportional to $\delta(0)$. By rearranging the terms such that the structure functions are on the left side and using the Jacobi identity (2.2) we obtain

$$\left[G^I, G^J\right] = G^K C_K{}^{IJ} = (-1)^{K(I+J+1)} C_K{}^{IJ} G^K\,, \tag{B.13}$$

   without any ordering terms. The second equality in (B.13) holds due to point 2.

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